
Heights and measures on analytic spaces.

A survey of recent results, and some remarks

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The first goal of this paper was to survey my definition in [19] of measures on non-archimedean analytic spaces in the sense of Berkovich and to explain its applications in Arakelov geometry. These measures are analogous the measures on complex analytic spaces given by products of first Chern forms of hermitian line bundles.⁽¹⁾ In both contexts, archimedean and non-archimedean, they are related with Arakelov geometry and the local height pairings of cycles. However, while the archimedean measures lie at the ground of the definition of the archimedean local heights in Arakelov geometry, the situation is reversed in the ultrametric case: we begin with the definition of local heights given by arithmetic intersection theory and define measures in such a way that the archimedean formulae make sense and are valid. The construction is outlined in Section 1, with references concerning metrized line bundles and the archimedean setting. More applications to Arakelov geometry and equidistribution theorems are discussed in Section 3.

The relevance of Berkovich spaces in Diophantine geometry has now made been clear by many papers; besides [19] and [20] and the general equidistribution theorem of Yuan [59], I would like to mention the works [38, 39, 40, 30] who discuss the function field case of the equidistribution theorem, as well as the potential theory on non-archimedean curves developed simultaneously by Favre, Jonsson & Rivera-Letelier [32, 33] and Baker & Rumely for the projective line [8], and in general by A. Thuillier's PhD thesis [55]. The reader will find many important results in the latter work, which unfortunately is still unpublished at the time of this writing.

1. M. Kontsevich and Yu. Tschinkel gave me copies of unpublished notes from the years 2000–2002 where they develop similar ideas to construct canonical non-archimedean metrics on Calabi–Yau varieties ; see also [45, 46].

Anyway, I found useful to add examples and complements to the existing (and non-) literature. This is done in Section 2. Especially, I discuss in Section 2.2 the relation between the reduction graph and the skeleton of a Berkovich curve, showing that the two constructions of measures coincide. Section 2.3 shows that the measures defined are of a local nature; more generally, we show that the measures vanish on any open subset where one of the metrized line bundles involved is trivial. This suggests a general definition of *strongly pluriharmonic* functions on Berkovich spaces, as uniform limits of logarithms of absolute values of invertible holomorphic functions. (Strongly pluriharmonic functions should only exhaust pluriharmonic functions when the residue field is algebraic over a finite field, but not in general.) In Section 2.4, we discuss polarized dynamical systems and explain the construction of canonical metrics and measures in that case. We also show that the canonical measure vanishes on the Berkovich equicontinuity locus. In fact, what we show is that the canonical metric is “strongly pluriharmonic” on that locus. This is the direct generalization of a theorem of [52] for the projective line (see also [8] for an exposition); this generalizes also a theorem of [44] that Green functions are locally constant on the classical equicontinuity locus. As already were their proofs, mine is a direct adaptation of the proof of the complex case [43]. In Section 2.5, following Gubler [41], we finally describe the canonical measures in the case of abelian varieties.

In Section 3, we discuss applications of the measures in Diophantine geometry over global fields. Once definitions are recalled out in Section 3.1, we briefly discuss in Section 3.2 the relation between Mahler measures (*i.e.*, integration of Green functions against measures) and heights. In Section 3.3, we survey the equidistribution theorems for Galois orbits of points of “small height”, following the variational method of Szpiro–Ullmo–Zhang [54] and [59]. In fact, we describe the more general statement from [20]. Finally, Section 3.4 discusses positive lower bounds for heights on curves. This is inspired by recent papers [5, 49] but the method goes back to Mimar’s unpublished thesis [48]. A recent preprint [58] of Yuan and Zhang establishes a similar result in any dimension.

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1. Metrized line bundles and measures

1.1. Continuous metrics

1.1.1. DEFINITION. — Let X be a topological space together with a sheaf of local rings \mathcal{O}_X (“analytic functions”); let also \mathcal{C}_X be the sheaf of continuous functions on X . In analytic geometry, local functions have an absolute value which is a real valued continuous function, satisfying the triangle inequality. Let us thus assume that we have a morphism of sheaves $\mathcal{O}_X \rightarrow \mathcal{C}_X$, written $f \mapsto |f|$, such that $|fg| = |f||g|$, $|1| = 1$, and $|f + g| \leq |f| + |g|$.

A line bundle on (X, \mathcal{O}_X) is a sheaf L of \mathcal{O}_X -modules which is locally isomorphic to \mathcal{O}_X . In other words, X is covered by open sets U such that $\mathcal{O}_U \simeq L|_U$; such an isomorphism is equivalent to a non-vanishing section $\varepsilon_U \in \Gamma(U, L)$, also called a local frame of L .

If s is a section of a line bundle L on an open set U , the value of s at a point $x \in U$ is only well-defined as an element of the stalk $L(x)$, which is a $\kappa(x)$ -vector space of dimension 1. (Here, $\kappa(x)$ is the residue field of \mathcal{O}_X at x .) Prescribing a metric on L amounts to assigning, in a coherent way, the norms of these values. Formally, a *metric* on L is the datum, for any open set $U \subset X$ and any section $s \in \Gamma(U, L)$, of a continuous function $\|s\|_U : U \rightarrow \mathbf{R}_+$, satisfying the following properties:

- (1) for any open set $V \subset U$, $\|s\|_V$ is the restriction to V of the function $\|s\|_U$;
- (2) for any function $f \in \mathcal{O}_X(U)$, $\|fs\| = |f| \|s\|$;
- (3) if s is a local frame on U , then $\|s\|$ doesn’t vanish at any point of U .

One usually writes \overline{L} for the pair $(L, \|\cdot\|)$ of a line bundle L and a metric on it.

Observe that the trivial line bundle \mathcal{O}_X has a natural “trivial” metric, for which $\|1\| = 1$. In fact, a metric on the trivial line bundle \mathcal{O}_X is equivalent to the datum of a continuous function h on X , such that $\|1\| = e^{-h}$.

1.1.2. THE ABELIAN GROUP OF METRIZED LINE BUNDLES. — Isomorphism of metrized line bundles are isomorphisms of line bundles which respect the metrics; they are called *isometries*. Constructions from tensor algebra extend naturally to the framework of metrized line bundles, compatibly with isometries. The tensor product of two metrized line bundles \overline{L} and \overline{M} has a natural metrization such that $\|s \otimes t\| = \|s\| \|t\|$, if s and t are local sections of L and M respectively. Similarly, the dual of a metrized line bundle has a metrization, and the obvious isomorphism $L \otimes L^\vee \simeq \mathcal{O}_X$ is an isometry. Consequently, isomorphism classes of metrized line bundles on X form an Abelian group $\overline{\text{Pic}}(X)$. This group fits in an exact sequence

$$0 \rightarrow \mathcal{C}(X) \rightarrow \overline{\text{Pic}}(X) \rightarrow \text{Pic}(X),$$

where the first map associates to a real continuous function h on X the trivial line bundle endowed with the metric such that $\|1\| = e^{-h}$, and the second associates to a metrized line bundle the underlying line bundle. It is surjective when any line bundle has a metric (this certainly holds if X has partitions of unity).

Similarly, we can consider pull-backs of metrized line bundle. Let $\varphi: Y \rightarrow X$ be a morphism of locally ringed spaces such that $|\varphi^* f| = |f| \circ \varphi$ for any $f \in \mathcal{O}_X$. Let \overline{L}

be a metrized line bundle on X . Then, there is a canonical metric on φ^*L such that $\|\varphi^*s\| = \|s\| \circ \varphi$ for any section $s \in \Gamma(U, L)$. This induces a morphism of Abelian groups $\varphi^*: \overline{\text{Pic}}(X) \rightarrow \overline{\text{Pic}}(Y)$.

1.2. The case of complex analytic spaces

1.2.1. SMOOTH METRICS. — In complex analytic geometry, metrics are a very well established tool. Let us first consider the case of the projective space $X = \mathbf{P}^n(\mathbf{C})$; a point $x \in X$ is a $(n+1)$ -tuple of homogeneous coordinates $[x_0 : \dots : x_n]$, not all zero, and up to a scalar. Let $\pi: \mathbf{C}_*^{n+1} \rightarrow X$ be the canonical projection map, where the index $*$ means that we remove the origin $(0, \dots, 0)$. The fibers of π have a natural action of \mathbf{C}^* and the line bundle $\mathcal{O}(1)$ has for sections s over an open set $U \subset \mathbf{P}^n(\mathbf{C})$ the analytic functions F_s on the open set $\pi^{-1}(U) \subset \mathbf{C}_*^{n+1}$ which are homogeneous of degree 1. The *Fubini-Study metric* of $\mathcal{O}(1)$ assigns to the section s the norm $\|s\|_{\text{FS}}$ defined by

$$\|s\|_{\text{FS}}([x_0 : \dots : x_n]) = \frac{|F_s(x_0, \dots, x_n)|}{(|x_0|^2 + \dots + |x_n|^2)^{1/2}}.$$

It is more than continuous; indeed, if s is a local frame on an open set U , then $\|s\|$ is a \mathcal{C}^∞ -function on U ; such metrics are called *smooth*.

1.2.2. CURVATURE. — Line bundles with smooth metrics on smooth complex analytic spaces allow to perform differential calculus. Namely, the *curvature* of a smooth metrized line bundle \bar{L} is a differential form $c_1(\bar{L})$ of type $(1, 1)$ on X . Its definition involves the differential operator

$$\text{dd}^c = \frac{i}{\pi} \partial \bar{\partial}.$$

When an open set $U \subset X$ admits local coordinates (z_1, \dots, z_n) , and $s \in \Gamma(U, L)$ is a local frame, then

$$c_1(\bar{L})|_U = \text{dd}^c \log \|s\|^{-1} = \frac{i}{\pi} \sum_{1 \leq j, k \leq n} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log \|s\|^{-1} dz_j \wedge d\bar{z}_k.$$

The Cauchy-Riemann equations ($\partial f / \partial \bar{z} = 0$ for any holomorphic function f of the variable z) imply that this formula does not depend on the choice of a local frame s . Consequently, these differential forms defined locally glue to a well-defined global differential form on X .

Taking the curvature form of a metrized line bundle is a linear operation: $c_1(\bar{L} \otimes \bar{M}) = c_1(\bar{L}) + c_1(\bar{M})$. It also commutes with pull-back: if $f: Y \rightarrow X$ is a morphism, then $f^* c_1(\bar{L}) = c_1(f^* \bar{L})$.

In the case of the Fubini-Study metric over the projective space $\mathbf{P}^n(\mathbf{C})$, the curvature is computed as follows. The open subset U_0 where the homogeneous coordinate x_0 is non-zero has local coordinates $z_1 = x_1/x_0, \dots, z_n = x_n/x_0$; the homogeneous polynomial X_0 defines a non-vanishing section s_0 of $\mathcal{O}(1)$ on U_0 and

$$\log \|s_0\|_{\text{FS}}^{-1} = \frac{1}{2} \log \left(1 + \sum_{j=1}^n |z_j|^2 \right).$$

Consequently, over U_0 ,

$$\begin{aligned} c_1(\overline{\mathcal{O}(1)}_{\text{FS}}) &= \frac{i}{\pi} \partial \bar{\partial} \log \|s_0\|_{\text{FS}}^{-1} \\ &= \frac{i}{2\pi} \partial \left(\sum_{k=1}^n \frac{z_k}{1 + \|z\|^2} d\bar{z}_k \right) \\ &= \frac{i}{2\pi} \sum_{j=1}^n \frac{1}{1 + \|z\|^2} dz_j \wedge d\bar{z}_j - \frac{i}{2\pi} \sum_{j,k=1}^n \frac{z_k \bar{z}_j}{(1 + \|z\|^2)^2} dz_j \wedge d\bar{z}_k. \end{aligned}$$

In this calculation, we have abbreviated $\|z\|^2 = \sum_{j=1}^n |z_j|^2$.

1.2.3. PRODUCTS, MEASURES. — Taking the product of n factors equal to this differential form, we get a differential form of type (n, n) on the n -dimensional complex space X . Such a form can be integrated on X and the Wirtinger formula asserts that

$$\int_X c_1(\bar{L})^n = \deg(L)$$

is the *degree* of L as computed by intersection theory. As an example, if $X = \mathbf{P}^1(\mathbf{C})$, we have seen that

$$c_1(\overline{\mathcal{O}(1)}_{\text{FS}}) = \frac{i}{2\pi(1 + |z|^2)^2} dz \wedge d\bar{z},$$

where $z = x_1/x_0$ is the affine coordinate of $X \setminus \{\infty\}$. Passing in polar coordinates $z = re^{i\theta}$, we get

$$c_1(\overline{\mathcal{O}(1)}_{\text{FS}}) = \frac{1}{2\pi(1 + r^2)^2} 2r dr \wedge d\theta$$

whose integral over \mathbf{C} equals

$$\int_{\mathbf{P}^1(\mathbf{C})} c_1(\overline{\mathcal{O}(1)}_{\text{FS}}) = \int_0^\infty \frac{1}{2\pi(1 + r^2)^2} 2r dr \int_0^{2\pi} d\theta = \int_0^\infty \frac{1}{(1 + u)^2} du = 1.$$

1.2.4. THE POINCARÉ–LELONG EQUATION. — An important formula is the Poincaré–Lelong equation. For any line bundle L with a smooth metric, and any section $s \in \Gamma(X, L)$ which does not vanish identically on any connected component of X , it asserts the following equality of *currents*⁽²⁾:

$$dd^c \log \|s\|^{-1} + \delta_{\text{div}(s)} = c_1(\bar{L}),$$

where $dd^c \log \|s\|^{-1}$ is the image of $\log \|s\|^{-1}$ under the differential operator dd^c , taken in the sense of distributions, and $\delta_{\text{div}(s)}$ is the current of integration on the cycle $\text{div}(s)$ of codimension 1.

2. The space of currents is the dual to the space of differential forms, with the associated grading; in the orientable case, currents can also be seen as differential forms with distribution coefficients.

1.2.5. ARCHIMEDEAN HEIGHT PAIRING. — Metrized line bundles and their associated curvature forms are a basic tool in Arakelov geometry, invented by Arakelov in [2] and developed by Faltings [31], Deligne [25] for curves, and by Gillet-Soulé [34] in any dimension. For our concerns, they allow for a definition of height functions for algebraic cycles on algebraic varieties defined over number fields. As explained by Gubler [35, 36], they also permit to develop a theory of archimedean local heights.

For simplicity, let us assume that X is proper, smooth, and that all of its connected components have dimension n .

Let $\bar{L}_0, \dots, \bar{L}_n$ be metrized line bundles with smooth metrics. For $j \in \{0, \dots, n\}$, let s_j be a regular meromorphic section of L_j and let $\text{div}(s_j)$ be its divisor. The given metric of L_j furnishes moreover a function $\log \|s_j\|^{-1}$ on X and a $(1, 1)$ -form $c_1(\bar{L}_j)$, related by the Poincaré–Lelong equation $\text{dd}^c \log \|s_j\|^{-1} + \delta_{\text{div}(s_j)} = c_1(\bar{L}_j)$. In the terminology of Arakelov geometry, $\log \|s_j\|^{-1}$ is a Green current (here, function) for the cycle $\text{div}(s_j)$; we shall write $\widehat{\text{div}}(s_j)$ for the pair $(\text{div}(s_j), \log \|s_j\|^{-1})$.

Let $Z \subset X$ be a k -dimensional subvariety such that the divisors $\text{div}(s_j)$, for $0 \leq j \leq k$, have no common point on Z . Then, one defines inductively the local height pairing by the formula:

$$(1.2.6) \quad (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) = (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_{k-1})| \text{div}(s_k|_Z)) \\ + \int_X \log \|s_k\|^{-1} c_1(\bar{L}_0) \dots c_1(\bar{L}_{k-1}) \delta_Z.$$

The second hand of this formula requires two comments. 1) The divisor $\text{div}(s_k|_Z)$ is a formal linear combination of $(k-1)$ -dimensional subvarieties of X , and its local height pairing is computed by linearity from the local height pairings of its components. 2) The integral of the right hand side involves a function with singularities ($\log \|s_k\|^{-1}$) to be integrated against a distribution: in this case, this means restricting the differential form $c_1(\bar{L}_0) \dots c_1(\bar{L}_{k-1})$ to the smooth part of Z , multiplying by $\log \|s_k\|^{-1}$, and integrating the result. The basic theory of closed positive currents proves that the resulting integral converges absolutely; as in [34], one can also resort to Hironaka’s resolution of singularities.

It is then a non-trivial result that the local height pairing is symmetric in the involved $\widehat{\text{divisors}}$; it is also multilinear. See [37] for more details, as well as [34] for the global case.

1.2.7. POSITIVITY. — Consideration of the curvature allows to define positivity notions for metrized line bundles. Namely, one says that a smooth metrized line bundle \bar{L} is *positive* (resp. *semi-positive*) if its curvature form is a positive (resp. a non-negative) $(1, 1)$ -form. This means that for any point $x \in X$, the hermitian form $c_1(\bar{L})_x$ on the complex tangent space $T_x X$ is positive definite (resp. non-negative). As a crucial example, the line bundle $\mathcal{O}(1)$ with its Fubini-Study metric is positive. The pull-back of a positive metrized line bundle by an immersion is positive. In particular, ample line bundles can be endowed with a positive smooth metric; Kodaira’s embedding theorem asserts the converse: if a line bundle possesses a positive smooth metric, then it is ample.

The pull-back of a semi-positive metrized line bundle by any morphism is still semi-positive. If \bar{L} is semi-positive, then the measure $c_1(\bar{L})^n$ is a positive measure.

1.2.8. SEMI-POSITIVE CONTINUOUS METRICS. — More generally, both the curvature and the Poincaré–Lelong equation make sense for metrized line bundles with arbitrary (continuous) metrics, except that $c_1(\bar{L})$ has to be considered as a current. The notion of semi-positivity can even be extended to this more general case, because it can be tested by duality: a current is positive if its evaluation on any nonnegative differential form is nonnegative. Alternatively, semi-positive (continuous) metrized line bundles are characterized by the fact that for any local frame s of \bar{L} over an open set U , the continuous function $\log \|s\|^{-1}$ is *plurisubharmonic* on U . In turn, this means that for any morphism $\varphi: \bar{D} \rightarrow U$, where $\bar{D} = \bar{D}(0, 1)$ is the closed unit disk in \mathbf{C} ,

$$\log \|s\|^{-1}(\varphi(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} \log \|s\|^{-1}(\varphi(e^{i\theta})) d\theta.$$

Assume that \bar{L} is semi-positive. Although products of currents are not defined in general (not more than products of distributions), the theory of Bedford–Taylor [10, 9] and Demailly [26, 27] defines a current $c_1(\bar{L})^n$ which then is a positive measure on X . There are two ways to define this current. The first one works locally and proceeds by induction: if $u = \log \|s\|^{-1}$, for a local non-vanishing section s of L , one defines a sequence (T_k) of closed positive currents by the formulae $T_0 = 1$, $T_1 = dd^c u, \dots$, $T_{k+1} = dd^c(uT_k)$ and $c_1(\bar{L})^n = dd^c(u)^n$ is defined to be T_n . What makes this construction work is the fact that at each step, uT_k is a well-defined current (product of a continuous function and of a positive current), and one has to prove that T_{k+1} is again a closed positive current. The other way, which shall be the one akin to a generalization in the ultrametric framework, consists in observing that if L is a line bundle with a continuous semi-positive metric $\|\cdot\|$, then there exists a sequence of smooth semi-positive metrics $\|\cdot\|_k$ on the line bundle L which converges uniformly to the initial metric: for any local section s , $\|s\|_k$ converges uniformly to $\|s\|$ on compact sets. The curvature current $c_1(\bar{L})$ is then the limit of the positive currents $c_1(\bar{L}_k)$, and the measure $c_1(\bar{L})^n$ is the limit of the measures $c_1(\bar{L}_k)^n$. (We refer to [47] for the global statement; to construct the currents, one can in fact work locally in which case a simple convolution argument establishes the claim.)

An important example of semi-positive metric which is continuous, but not smooth, is furnished by the Weil metric on the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^n(\mathbf{C})$. This metric is defined as follows: if $U \subset \mathbf{P}^n(\mathbf{C})$ is an open set, and s is a section of $\mathcal{O}(1)$ on U corresponding to an analytic function F_s on $\pi^{-1}(U) \subset \mathbf{C}_*^{n+1}$ which is homogeneous of degree 1, then for any $(x_0, \dots, x_n) \in \pi^{-1}(U)$, one has

$$\|s\|_W = \frac{|F_s(x_0, \dots, x_n)|}{\max(|x_0|, \dots, |x_n|)}.$$

The associated measure $c_1(\overline{\mathcal{O}(1)_W})^n$ on $\mathbf{P}^n(\mathbf{C})$ is as follows, cf. [62, 47]: the subset of all points $[x_0 : \dots : x_n] \in \mathbf{P}^n(\mathbf{C})$ such that $|x_j| = |x_k|$ for all j, k is naturally identified with the

polycircle \mathbf{S}_1^n (map $[x_0 : \dots : x_n]$ to $(x_1/x_0, \dots, x_n/x_0)$); take the normalized Haar measure of this compact group and push it onto $\mathbf{P}^n(\mathbf{C})$.

1.2.9. ADMISSIBLE METRICS. — Let us say that a continuous metrized line bundle is *admissible* if it can be written as $\bar{L} \otimes \bar{M}^\vee$, where \bar{L} and \bar{M} are metrized line bundles whose metrics are continuous and semi-positive. Admissible metrized line bundles form a subgroup $\overline{\text{Pic}}_{\text{ad}}(X)$ of $\overline{\text{Pic}}(X)$ which maps surjectively onto $\text{Pic}(X)$ if X is projective.

The curvature current $c_1(\bar{L})$ of an admissible metrized line bundle \bar{L} is a differential form of type $(1, 1)$ whose coefficients are signed measures. Its n th product $c_1(\bar{L})^n$ is well-defined as a signed measure on X .

1.2.10. LOCAL HEIGHT PAIRING (ADMISSIBLE CASE). — The good analytic properties of semi-positive metrics allow to extend the definition of the local height pairing to the case of admissible line bundles. Indeed, when one approximates uniformly a semi-positive line bundle by a sequence of smooth semi-positive line bundles, one can prove that the corresponding sequence of local height pairings converges, the limit being independent on the chosen approximation.

The proof is inspired by Zhang's proof of the global case in [63] and goes by induction. Let us consider, for each j , two smooth semi-positive metrics on the line bundle L_j and assume that they differ by a factor e^{-h_j} . Then, the corresponding local height pairings differ from an expression of the form

$$\sum_{j=0}^k \int_Z h_j c_1(\bar{L}_0) \dots \widehat{c_1(\bar{L}_j)} \dots c_1(\bar{L}_k),$$

where the written curvature forms are associated to the first metric for indices $< j$, and to the second for indices $> j$. This differential forms are positive by assumption, so that the integral is bounded in absolute value by

$$\begin{aligned} \sum_{j=0}^k \|h_j\|_\infty \int_Z c_1(\bar{L}_0) \dots \widehat{c_1(\bar{L}_j)} \dots c_1(\bar{L}_k) \\ = \sum_{j=0}^K \|h_j\|_\infty (c_1(L_0) \dots \widehat{c_1(L_j)} \dots c_1(L_k)|Z), \end{aligned}$$

where the last expression is essentially a degree. (In these formulae, the factor with a hat is removed.) This inequality means that on the restriction to the space of smooth semi-positive metrics, with the topology of uniform convergence, the local height pairing is uniformly continuous. Therefore, it first extends by continuity on the space of continuous semi-positive metrics, and then by multilinearity to the space of admissible metrics.

1.3. The case of non-archimedean analytic spaces

Let K be a complete ultrametric field. We are principally interested in finite extensions of \mathbf{Q}_p , but the case of local fields of positive characteristic (finite extensions of $k((T))$, for a finite field k) have proved being equally useful, as are non-local fields

like the field $\mathbf{C}((T))$ of Laurent power series with *complex* coefficients. For simplicity, we will assume that K is the field of fractions of a complete discrete valuation ring K° , let π be a generator of the maximal ideal of K° and let $\tilde{K} = K^\circ/(\pi)$ be the residue field.

1.3.1. CONTINUOUS METRICS. — Let X be a K -analytic space in the sense of Berkovich [12]. For simplicity, we will assume that X is the analytic space associated to a *proper* scheme over K . In that context, the general definition of continuous metrized line bundles given above makes sense.

Let us detail the example of the line bundle $\mathcal{O}(1)$ on the projective space \mathbf{P}_K^n . A point $x \in \mathbf{P}_K^n$ possesses a complete residue field $\mathcal{H}(x)$ which is a complete extension of K and homogeneous coordinates $[x_0 : \dots : x_n]$ in the field $\mathcal{H}(x)$. As in complex geometry, the projective space \mathbf{P}_K^n is obtained by glueing $n+1$ copies U_0, \dots, U_n of the affine space A_K^n , where U_i corresponds to those points x such that $x_i \neq 0$. Recall also that A_K^n is the space of multiplicative semi-norms on the K -algebra $K[T_1, \dots, T_n]$ which induce the given absolute value on K , together with the coarsest topology such that for any semi-norm $x \in A_K^n$, the map $K[T_1, \dots, T_n] \rightarrow \mathbf{R}$ defined by $f \mapsto x(f)$ is continuous. The kernel of a semi-norm x is a prime ideal \mathfrak{p}_x of $K[T_1, \dots, T_n]$ and x induces a norm on the quotient ring $K[T_1, \dots, T_n]/\mathfrak{p}_x$, hence on its field of fractions $K(x)$. The completion of $K(x)$ with respect to this norm is denoted $\mathcal{H}(x)$ and is called the *complete residue field* of x . The images in $\mathcal{H}(x)$ of the indeterminates T_i are denoted $T_i(x)$, more generally, the image in $\mathcal{H}(x)$ of any polynomial $f \in K[T_1, \dots, T_n]$ is denoted $f(x)$; one has $x(f) = |f(x)|$.

Let f be a rational function on \mathbf{P}_K^n , that is an element of $K(T_1, \dots, T_n)$. It defines an actual function on the open set U of \mathbf{P}_K^n where its denominator does not vanish; its value at a point $x \in U$ is an element of $\mathcal{H}(x)$. More generally, Berkovich defines an analytic function on an open set U of \mathbf{P}_K^n as a function f on U such that $f(x) \in \mathcal{H}(x)$ for any $x \in U$, and such that any point $x \in U$ possesses a neighbourhood $V \subset U$ such that $f|_V$ is a uniform limit of rational functions without poles on V .

The line bundle $\mathcal{O}(1)$ can also be defined in a similar way to the classical case; by a similar GAGA theorem, its global sections are exactly the same as in algebraic geometry and are described by homogeneous polynomials of degree 1 with coefficients in K . If P is such a polynomial and s_P the corresponding section, then

$$\|s_P\|(x) = \frac{|P(x_0, \dots, x_n)|}{\max(|x_0|, \dots, |x_n|)}$$

where $[x_0 : \dots : x_n]$ is a system of homogeneous coordinates in $\mathcal{H}(x)$ for the point x . The function $\|s_P\|$ is continuous on \mathbf{P}_K^n , by the very definition of the topology on \mathbf{P}_K^n . Using the fact that $\mathcal{O}(1)$ is generated by its global sections, one deduces the existence of a continuous metric on $\mathcal{O}(1)$ satisfying the previous formula.

1.3.2. SMOOTH METRICS. — Following [63], we now want to explain the analogues of smooth, and, later, of semi-positive metrics.

Smooth metrics come from algebraic geometry over K° , and, more generally, over the ring of integers of finite extensions of K . Let namely \mathfrak{X} be a formal proper K^0 -scheme whose generic fibre in the sense of analytic geometry is X .⁽³⁾ Let also \mathcal{L} be a line bundle on \mathfrak{X} which is model of some power L^e , where $e \geq 1$. From this datum $(\mathfrak{X}, \mathcal{L}, e)$, we can define a metric on L as follows. Let \mathfrak{U} be a formal open subset of \mathfrak{X} over which \mathcal{L} admits a local frame $\varepsilon_{\mathfrak{U}}$; over its generic fibre $U = \mathfrak{U}_K$, for any section s of L , one can write canonically $s^e = f \varepsilon_{\mathfrak{U}}$, where $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$. We decree that $\|s\| = |f|^{1/e}$. In other words, the norm of a local frame on the formal model is assigned to be identically one. This makes sense because if $\eta_{\mathfrak{U}}$ is another local frame of \mathcal{L} on \mathfrak{U} , there exists an invertible formal function $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})^*$ such that $\eta_{\mathfrak{U}} = f \varepsilon_{\mathfrak{U}}$ and the absolute value $|f|$ of the associated analytic function on U is identically equal to 1. Considering a finite cover of \mathfrak{X} by formal open subsets, their generic fibers form a finite cover of X by *closed* subsets and this is enough to glue the local definitions to a continuous metric on L .

Metrics on L given by this construction, for some model $(\mathfrak{X}, \mathcal{L}, e)$ of some power L^e of L will be said to be *smooth*.

1.3.3. GREEN FUNCTIONS; SMOOTH FUNCTIONS. — Let \overline{L} be a metrized line bundle and let s be a regular meromorphic section of L . Its divisor $\text{div}(s)$ is a Cartier divisor in X . The function $\log \|s\|^{-1}$ is defined on the open set $X \setminus |\text{div}(s)|$; by analogy to the complex case, we call it a *Green function* for the divisor $\text{div}(s)$. When the metric on \overline{L} is smooth, the Green function is said to be smooth. The same remark applies for the other qualificatives semi-positive, or admissible, that will be introduced later.

Let us take for L the trivial line bundle, with its canonical trivialization $s = 1$, and let us endow it with a smooth metric. By definition, we call $\log \|s\|^{-1}$ a smooth function. More generally, we define the space $\mathcal{C}^\infty(X)$ of (real valued) smooth functions to be the real vector space spanned by these elementary smooth functions. Observe that this definition reverses what happens in complex geometry where smooth metrics on the trivial line bundle are *defined* from the knowledge of smooth functions.

1.3.4. EXAMPLE: PROJECTIVE SPACE. — Let us consider the smooth metric on $\mathcal{O}(1)$ associated to the model $(\mathbf{P}_{K^\circ}^n, \mathcal{O}(1), 1)$ of $(\mathbf{P}_K^n, \mathcal{O}(1))$. Let \mathfrak{U}_i be the formal open subset of $\mathbf{P}_{K^\circ}^n$ defined by the non-vanishing of the homogeneous coordinate x_i . Over, $\mathfrak{U}_i, \mathcal{O}(1)$ has a global non-vanishing section, namely the one associated to the homogeneous polynomial X_i . The generic fiber U_i of \mathfrak{U}_i in the sense of algebraic geometry is an affine space, with coordinates $z_j = x_j/x_i$, for $0 \leq j \leq n$, and $j \neq i$. However, its generic fiber U_i in the sense of rigid geometry is the n -dimensional polydisk in this affine space defined by the inequalities $|z_j| \leq 1$. We thus observe that for any $x \in (\mathfrak{U}_i)_K$,

$$\begin{aligned} \|X_i\|(x) &= 1 \\ &= \frac{1}{\max(|z_0|, \dots, |z_{i-1}|, 1, |z_{i+1}|, \dots, |z_n|)} \end{aligned}$$

3. The reader might want to assume that X is the analytic space associated to a projective K -scheme X and that \mathfrak{X} is a projective K^0 -scheme whose generic fibre equals X . This doesn't make too much of a difference for our concerns.

$$\begin{aligned}
&= \frac{|x_i|}{\max(|x_0|, \dots, |x_i|)} \\
&= \|X_i\|_W(x).
\end{aligned}$$

In other words, the Weil metric on $\mathcal{O}(1)$ is a smooth metric.

1.3.5. THE ABELIAN GROUP OF SMOOTH LINE BUNDLES. — Let us show that *any line bundle has a smooth metric*. There is a general theory, due to Raynaud, that shows how to define formal models from rigid analytic objects. In the present case, X being projective, we may assume that L is ample and consider a closed embedding of X in a projective space \mathbb{P}_K^n given by some power L^e . Let \mathfrak{X} be the Zariski closure of X in $\mathbb{P}_{K^\circ}^n$; in concrete terms, if $I \subset K[X_0, \dots, X_n]$ is the homogeneous ideal of $i(X)$, $I \cap K^\circ[X_0, \dots, X_n]$ is the homogeneous ideal of \mathfrak{X} . Let then \mathfrak{L} be the restriction to \mathfrak{X} of the line bundle $\mathcal{O}(1)$. The triple $(\mathfrak{X}, \mathfrak{L}, e)$ is a model of L and induces a smooth metric on L .

Different models can give rise to the same metric. If $\varphi: \mathfrak{X}' \rightarrow \mathfrak{X}$ is a morphism of models, and $\mathfrak{L}' = \varphi^* \mathfrak{L}$, then $(\mathfrak{X}', \mathfrak{L}', e)$ defines the same smooth metric on L . Moreover, if two models $(\mathfrak{X}_i, \mathfrak{L}_i, e_i)$, for $i \in \{1, 2\}$, define the same metric, there exists a third model $(\mathfrak{X}, \mathfrak{L})$, with two morphisms $\varphi_i: \mathfrak{X} \rightarrow \mathfrak{X}_i$ such that the pull-backs $\varphi_i^* \mathfrak{L}_i^{e_1 e_2 / e_i}$ coincide with \mathfrak{L} . More precisely, if two models \mathfrak{L} and \mathfrak{L}' of some power L^e on a *normal* model \mathfrak{X} define the same metric, then they are isomorphic. (See, e.g., Lemma 2.2 of [20]; this may be false for non-normal models; it suffices that \mathfrak{X} be integrally closed in its generic fiber.)

As a consequence, the set $\overline{\text{Pic}}_{\text{sm}}(X)$ of smooth metrized line bundles is a subgroup of the group $\overline{\text{Pic}}(X)$. The group $\overline{\text{Pic}}_{\text{sm}}(X)$ fits within an exact sequence

$$0 \rightarrow \mathcal{C}^\infty(X) \rightarrow \overline{\text{Pic}}_{\text{sm}}(X) \rightarrow \text{Pic}(X) \rightarrow 0,$$

the last map is surjective because every line bundle admits a model. If $f: Y \rightarrow X$ is a morphism, then $f^*(\overline{\text{Pic}}_{\text{sm}}(X)) \subset \overline{\text{Pic}}_{\text{sm}}(Y)$.

1.3.6. SEMI-POSITIVE METRICS. — A smooth metric is said to be *ample* if it is defined by a model $(\mathfrak{X}, \mathfrak{L}, e)$ such that the restriction $\mathfrak{L}_{\tilde{K}}$ of \mathfrak{L} to the closed fiber $\mathfrak{X}_{\tilde{K}}$ is ample. The Weil metric on the line bundle $\mathcal{O}(1)$ on the projective space is ample. The proof given above of the existence of smooth metrics shows, more precisely, that ample line bundles admit ample metrics, and that *the pull-back of a smooth ample metric by an immersion is a smooth ample metric*.

A smooth metric is said to be *semi-positive* if it can be defined on a model $(\mathfrak{X}, \mathfrak{L}, e)$ such that the restriction $\mathfrak{L}_{\tilde{K}}$ of \mathfrak{L} to the closed fiber $\mathfrak{X}_{\tilde{K}}$ is numerically effective: for any projective curve $C \subset \mathfrak{X}_{\tilde{K}}$, the degree of the restriction to C of $\mathfrak{L}_{\tilde{K}}$ is non-negative. Ample metrics are semi-positive.

The pull-back of a smooth semi-positive metric by any morphism is semi-positive.

1.3.7. CONTINUOUS SEMI-POSITIVE METRICS. — Let us say that a continuous metric on a line bundle L is semi-positive if it is the uniform limit of a sequence of smooth semi-positive metrics on the same line bundle L . As in the complex case, we then say that a metrized line bundle is *admissible* if it can be written as $\overline{L} \otimes \overline{M}^\vee$, for two line bundles L and M with continuous semi-positive metrics.

Let L be a metrized line bundle, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two continuous metrics on L . It follows from the definition that the metrics $\|\cdot\|_{\min} = \min(\|\cdot\|_1, \|\cdot\|_2)$ and $\|\cdot\|_{\max} = \max(\|\cdot\|_1, \|\cdot\|_2)$ are continuous metrics.

Moreover, *these metrics $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ are smooth if the initial metrics are smooth.* Indeed, there exists a model \mathfrak{X} , as well as two line bundles \mathfrak{L}_1 and \mathfrak{L}_2 extending the same power L^ρ of L and defining the metrics $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. We may assume that \mathfrak{L}_1 and \mathfrak{L}_2 have regular global sections s_1 and s_2 on \mathfrak{X} which coincide on X , with divisors \mathfrak{D}_1 and \mathfrak{D}_2 respectively. (The general case follows, by twisting \mathfrak{L}_1 and \mathfrak{L}_2 by a sufficiently ample line bundle on \mathfrak{X} .) The blow-up $\pi: \mathfrak{X}' \rightarrow \mathfrak{X}$ of the ideal $\mathfrak{I}_{\mathfrak{D}_1} + \mathfrak{I}_{\mathfrak{D}_2}$ carries an invertible ideal sheaf $\mathfrak{I}_{\mathfrak{E}} = \pi^*(\mathfrak{I}_{\mathfrak{D}_1} + \mathfrak{I}_{\mathfrak{D}_2})$, with corresponding Cartier divisor \mathfrak{E} . Since \mathfrak{D}_1 and \mathfrak{D}_2 coincide on the generic fiber, $\mathfrak{I}_{\mathfrak{D}_1} + \mathfrak{I}_{\mathfrak{D}_2}$ is already invertible there and π is an isomorphism on the generic fiber.

The divisors $\pi^*\mathfrak{D}_1$ and $\pi^*\mathfrak{D}_2$ decompose canonically as sums

$$\pi^*\mathfrak{D}_1 = \mathfrak{D}'_1 + \mathfrak{E}, \quad \pi^*\mathfrak{D}_2 = \mathfrak{D}'_2 + \mathfrak{E}.$$

Let us pose

$$\mathfrak{D}' = \mathfrak{D}'_1 + \mathfrak{D}'_2 + \mathfrak{E} = \mathfrak{D}'_1 + \pi^*\mathfrak{D}_2 = \pi^*\mathfrak{D}_1 + \mathfrak{D}'_2.$$

An explicit computation on the blow-up shows that $(\mathfrak{X}', \mathfrak{D}', e)$ and $(\mathfrak{X}', \mathfrak{E}, e)$ are models of $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ respectively. In particular, these metrics are smooth.

Assume that the initial metrics are semi-positive, and that some positive power of L is effective. Then, the metric $\|\cdot\|_{\min}$ is semi-positive too. By approximation, it suffices to treat the case where the initial metrics are smooth and semi-positive. Then, the previous construction applies. Keeping the introduced notation, let us show that the restriction to the special fiber of the divisor $(\mathfrak{D}')_{\bar{K}}$ is numerically effective. Let $C \subset \mathfrak{X}'_{\bar{K}}$ be an integral curve and let us prove that $C \cdot (\mathfrak{D}')_{\bar{K}}$ is nonnegative. If C is not contained in \mathfrak{D}'_1 , then $C \cdot (\mathfrak{D}'_1)_{\bar{K}} \geq 0$, and $C \cdot (\pi^*\mathfrak{D}_2)_{\bar{K}} = \pi_* C \cdot \mathfrak{D}_2 \geq 0$ since $(\mathfrak{D}_2)_{\bar{K}}$ is numerically effective; consequently, $C \cdot (\mathfrak{D}')_{\bar{K}} \geq 0$. Similarly, $C \cdot (\mathfrak{D}')_{\bar{K}} \geq 0$ when C is not contained in \mathfrak{D}'_2 . Since $\mathfrak{D}'_1 \cap \mathfrak{D}'_2 = \emptyset$, this shows that $C \cdot \mathfrak{D}'_{\bar{K}} \geq 0$ in any case, hence $(\mathfrak{D}')_{\bar{K}}$ is numerically effective.

This last result is the analogue in the ultrametric case to the fact that the maximum of two continuous plurisubharmonic functions is continuous plurisubharmonic. However, observe that in the complex case, the maximum or the minimum of smooth functions are not smooth in general.

1.3.8. MEASURES (SMOOTH METRICS). — In the non-archimedean case, there isn't yet a purely analytic incarnation of the curvature form (or current) $c_1(\bar{L})$ of a metrized line bundle \bar{L} , although the non-archimedean Arakelov geometry of [14] should certainly be pushed forward in that direction. However, as I discovered in [19], one can define an analogue of the measure $c_1(\bar{L})^n$ when the space X has dimension n .

The idea consists in observing the local height pairing (defined by arithmetic intersection theory) and defining the measures so that a formula analogous to the complex one holds.

Let us therefore consider smooth metrized line bundles \bar{L}_j (for $0 \leq j \leq n$) as well as regular meromorphic sections s_j which have no common zero on X . There exists

a proper model \mathfrak{X} of X over K° and, for each j , a line bundle \mathfrak{L}_j on \mathfrak{X} which extends some power $L_j^{e_j}$ of L_j and which defines its metric.

Let $Z \subset X$ be an algebraic k -dimensional subvariety and let \mathfrak{Z} be its Zariski closure in \mathfrak{X} ; this is a $(k+1)$ -dimensional subscheme of \mathfrak{X} . Let's replace it by its normalization or, more precisely, by its integral closure in its generic fiber. The local height pairing is then given by intersection theory, as

$$(\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) = (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_k|_{\mathfrak{Z}}))|\mathfrak{Z}) \log|\pi|^{-1},$$

where $\text{div}(s_j|_{\mathfrak{Z}})$ means the divisor of s_j , viewed as a regular meromorphic section of \mathfrak{L}_j over \mathfrak{Z} . The right hand side means taking the intersection of the indicated Cartier divisors on \mathfrak{Z} , which is a well-defined class of a 0-cycle supported by the special fiber of \mathfrak{Z} ; then take its degree and multiply it by $\log|\pi|^{-1}$. (Recall that π is a fixed uniformizing element of K ; its absolute value does not depend on the actual choice.)

When one views $s_k|_Z$ as a regular meromorphic section of \mathfrak{L}_k on \mathfrak{Z} its divisor has two parts: the first one, say H , is “horizontal” and is the Zariski closure of the divisor $\text{div}(s_k|_Z)$; the second one, say V , is vertical, *i.e.*, lies in the special fiber of \mathfrak{Z} over the residue field of K° . This decomposes the local height pairing as a sum

$$\begin{aligned} (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) &= (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_k|_{\mathfrak{Z}}))|\mathfrak{Z}) \log|\pi|^{-1} \\ &= (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_{k-1}|_{\mathfrak{Z}}))|\text{div}(s_k|_{\mathfrak{Z}})) \log|\pi|^{-1} \\ &= (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_{k-1}|_{\mathfrak{Z}}))|H) \log|\pi|^{-1} \\ &\quad + (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_{k-1}|_{\mathfrak{Z}}))|V) \log|\pi|^{-1}. \end{aligned}$$

The first term is the local height pairing of $\text{div}(s_k|_Z)$. Let us investigate the second one.

Let (V_i) be the family of irreducible components of this special fiber; for each i , let m_i be its multiplicity in the fiber. Then, the vertical component V of $\text{div}(s_k|_{\mathfrak{Z}})$ decomposes as

$$V = \sum_i c_i m_i V_i,$$

where c_i is nothing but the order of vanishing of s_k along the special fiber at the generic point of V_i . Then,

$$\begin{aligned} (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_{k-1}|_{\mathfrak{Z}}))|V) &= \sum_i c_i m_i (c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_{k-1}|_{\mathfrak{Z}}))|V_i). \end{aligned}$$

Since V_i lies within the special fiber of \mathfrak{X} ,

$$(c_1(\text{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\text{div}(s_{k-1}|_{\mathfrak{Z}}))|V_i) = (c_1(\mathfrak{L}_0) \dots c_1(\mathfrak{L}_{k-1})|V_i),$$

the multidegree of the vertical component V_i with respect to the restriction on the special fiber of the line bundles $\mathfrak{L}_0, \dots, \mathfrak{L}_{k-1}$.

One remarkable aspect of Berkovich's theory is the existence, for each i , of a unique point v_i in Z which specializes to the generic point of V_i . (Here, we use that \mathfrak{Z} is integrally closed in its generic fibre.) Then,

$$\log \|s_k\|^{-1}(v_i) = c_i \log |\pi|^{-1}.$$

Finally,

$$\begin{aligned} (c_1(\operatorname{div}(s_0|_{\mathfrak{Z}})) \dots c_1(\operatorname{div}(s_{k-1}|_{\mathfrak{Z}}))|V) \log |\pi|^{-1} \\ = \sum_i \log \|s_k\|^{-1}(v_i) (c_1(\mathfrak{L}_0) \dots c_1(\mathfrak{L}_{k-1})|V_i). \end{aligned}$$

Let us sum up this calculation: we have introduced points $v_i \in Z$ and decomposed the local height pairing as a sum:

$$\begin{aligned} (\widehat{\operatorname{div}}(s_0) \dots \widehat{\operatorname{div}}(s_k)|Z) &= (\widehat{\operatorname{div}}(s_0) \dots \widehat{\operatorname{div}}(s_{k-1})|\operatorname{div}(s_k|_Z)) \\ &\quad + \sum_i \log \|s_k\|^{-1}(v_i) m_i(c_1(\mathfrak{L}_0) \dots c_1(\mathfrak{L}_{k-1})|V_i). \end{aligned}$$

It now remains to define

$$(1.3.9) \quad c_1(\bar{L}_0) \dots c_1(\bar{L}_{k-1})\delta_Z = \sum_i m_i(c_1(\mathfrak{L}_0) \dots c_1(\mathfrak{L}_{k-1})|V_i)\delta_{v_i},$$

where δ_{v_i} is the Dirac measure at the point $v_i \in Z$. This is a *measure* on X , whose support is contained in Z , and whose total mass equals

$$\begin{aligned} \sum_i (c_1(\mathfrak{L}_0) \dots c_1(\mathfrak{L}_{k-1})|V_i) &= (c_1(\mathfrak{L}_0) \dots c_1(\mathfrak{L}_{k-1})|V) \\ &= (c_1(L_0) \dots c_1(L_{k-1})|Z). \end{aligned}$$

One can also check that it does not depend on the choice of the section s_k .

With this definition, the local height pairing obeys an induction formula totally analogous to the one satisfied in the complex case:

$$\begin{aligned} (1.3.10) \quad (\widehat{\operatorname{div}}(s_0) \dots \widehat{\operatorname{div}}(s_k)|Z) \\ = (\widehat{\operatorname{div}}(s_0) \dots \widehat{\operatorname{div}}(s_{k-1})|\operatorname{div}(s_k|_Z)) \\ \quad + \int_X \log \|s_k\|^{-1} c_1(\bar{L}_0) \dots c_1(\bar{L}_{k-1})\delta_Z. \end{aligned}$$

1.3.11. LOCAL HEIGHT PAIRING (ADMISSIBLE METRICS). — With the notation of the previous paragraph, observe that the measures we have defined are positive when the smooth metrized line bundles are semi-positive. Indeed, this means that the line bundles \mathfrak{L}_j are numerically effective hence, as a consequence of the criterion Nakai–Moishezon, any subvariety of the special fiber has a nonnegative multidegree.

With basically the same argument that the one we sketched in the complex case, we conclude that the local height pairing extends by continuity when semi-positive metrized line bundles are approximated by smooth semi-positive metrized line bundles. By linearity, this extends the local height pairing to admissible metrized line bundles.

1.3.12. MEASURES (ADMISSIBLE METRICS). — Let us now return to semi-positive metrized line bundles $\bar{L}_0, \dots, \bar{L}_{k-1}$, approximated by smooth semi-positive metrized line bundles $\bar{L}_j^{(m)}$. I claim that for any k -dimensional variety $Z \subset X$, the measures $c_1(\bar{L}_0^{(m)}) \dots c_1(\bar{L}_{k-1}^{(m)}) \delta_Z$ converge to a measure on X .

To prove the claim, we may assume that $\bar{L}_0, \dots, \bar{L}_{k-1}$ have sections s_0, \dots, s_{k-1} without common zeroes on Z . Let also consider a smooth function φ on X ; let \bar{L}_k be the trivial line bundle with the section $s_k = 1$, metrized in such a way that $\|s_k\| = e^{-\varphi}$. Then, one has

$$\int_X \varphi c_1(\bar{L}_0^{(m)}) \dots c_1(\bar{L}_{k-1}^{(m)}) \delta_Z = (\widehat{\text{div}}(s_0)^{(m)} \dots \widehat{\text{div}}(s_{k-1})^{(m)} \widehat{\text{div}}(s_k)|Z);$$

writing \bar{L}_k has the quotient of two ample metrized line bundles, we deduce from the existence of the local height pairing for admissible metrics that these integrals converge when $m \rightarrow \infty$. Consequently, the sequence of measures $(c_1(\bar{L}_0^{(m)}) \dots c_1(\bar{L}_{k-1}^{(m)}) \delta_Z)_m$ converges to a positive linear form on the space of smooth functions. By a theorem of Gubler ([36], Theorem 7.12), which builds on the Stone-Weierstraß theorem and the compactness of the Berkovich space X , the space of smooth functions is dense in the space of continuous complex functions on X . A positivity argument, analogous to the proof that positive distributions are measures, then implies that our linear form is actually a positive measure which deserves the notation

$$c_1(\bar{L}_0) \dots c_1(\bar{L}_{k-1}) \delta_Z.$$

We then extend this definition by linearity to the case of arbitrary admissible line bundles. The total mass of this measure is again the multidegree of Z with respect to the line bundles L_j (for $0 \leq j \leq k-1$).

1.3.13. INTEGRATING GREEN FUNCTIONS. — The definition of the convergence of a sequence of measures is convergence of all integrals against a given *continuous* compactly supported function. In applications, however, it can be desirable to integrate against more general functions. The inductive formula (1.2.6) for the local height pairing in the complex case, is such an example, as is the interpretation of Mahler measures of polynomials as (the archimedean component of) heights. However, its analogue (Equation 1.3.10) a priori holds only when $\log \|s_0\|^{-1}$ is continuous, that is when the section s_0 has no zeroes nor poles.

The fact that it still holds in the archimedean case is a theorem of Maillot [47] building on the theory of Bedford–Taylor. We proved in [20, Th. 4.1] that this relation holds in the ultrametric case too. The proof (valid both in the ultrametric and archimedean cases) works by induction, and ultimately relies on an approximation lemma according to which any semi-positive Green function g for a divisor D is an increasing limit of smooth functions (g_n) such that, for any n , $g - g_n$ is a semi-positive Green function for D . In fact, it suffices to pose $g_n = \min(g, n \log |\pi|^{-1})$; then, $g - g_n = \max(0, g - n \log |\pi|^{-1})$ is the maximum of two semi-positive Green functions, hence is semi-positive. (In the archimedean case, one needs to further regularize g_n ; see [20] for details.)

The symmetry of the local height pairing then implies the following analogue of the Poincaré–Lelong formula. When \bar{L} is the trivial line bundle, with the metric defined by an admissible function φ , the factor $c_1(\bar{L})$ will be written $\text{dd}^c \varphi$, by analogy to the complex case.

1.3.14. Proposition. — *Let φ be a smooth function on X and let $\bar{L}_1, \dots, \bar{L}_k$ be admissible metrized line bundles; let Z be a k -dimensional subvariety of X and let s be an invertible meromorphic sections of \bar{L}_1 . Then,*

$$\begin{aligned} \int_X \varphi c_1(\bar{L}_1) \dots c_1(\bar{L}_k) \delta_Z \\ = \int_X \varphi c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_{\text{div}(s|Z)} + \int_X \log \|s\|^{-1} \text{dd}^c \varphi c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_Z. \end{aligned}$$

Proof. — Let \bar{L}_0 be the trivial line bundle with global section $s_0 = 1$ and metric defined by $\varphi = \log \|s_0\|^{-1}$. Let $s_1 = s$ and, for $2 \leq j \leq k$, let s_j be an invertible meromorphic section of L_j . Since $\text{div}(s_0|Z) = 0$,

$$(\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) = \int_X \varphi c_1(\bar{L}_1) \dots c_1(\bar{L}_k) \delta_Z$$

and

$$(\widehat{\text{div}}(s_0) \widehat{\text{div}}(s_2) \dots \widehat{\text{div}}(s_k)|\text{div}(s|Z)) = \int_X \varphi c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_{\text{div}(s|Z)}$$

One the other hand, the symmetry of the local height pairing implies that

$$\begin{aligned} (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) &= (\widehat{\text{div}}(s_1) \widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) \\ &= (\widehat{\text{div}}(s_0) \widehat{\text{div}}(s_2) \dots \widehat{\text{div}}(s_k)|\text{div}(s|Z)) \\ &\quad + \int_X \log \|s\|^{-1} c_1(\bar{L}_0) c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_Z. \end{aligned}$$

Combining these equations, we obtain the claim. \square

2. Examples

In this section, we give some examples of metrics and measures. Without mention of the contrary, we stick to the non-archimedean case; basic notation concerning K , K° , etc., is as in Section 1.3.

2.1. The projective space

Let X be the projective space \mathbf{P}_K^n and let $\mathcal{O}(1)$ be the tautological line bundle on X , together with its Weil metric. Let us describe the associated measure, taking the opportunity to add details concerning Berkovich spaces.

As we remarked above, the Weil metric is induced by the tautological line bundle on the projective scheme $\mathfrak{X} = \mathbf{P}_{K^\circ}^n$ and is smooth. The special fiber of \mathfrak{X} is the projective space $\mathbf{P}_{\bar{K}}^n$ over the residue field of K° ; it is in particular irreducible. Moreover, the degree of the tautological line bundle is equal to 1. The measure $c_1(\overline{\mathcal{O}(1)})^n$ is therefore equal

to the Dirac mass at the unique point of X which reduces to the generic point of the special fiber. It remains to describe this point more precisely.

The scheme $\mathbf{P}_{K^\circ}^n$ is the union of $(n+1)$ affine open subsets $\mathcal{U}_0, \dots, \mathcal{U}_n$ defined by the non-vanishing of the homogeneous coordinates x_0, \dots, x_n . Their generic fibers in the sense of analytic geometry are $n+1$ affinoid subsets U_0, \dots, U_n , which cover \mathbf{P}_K^n . In fact, U_i corresponds to the set of points $[x_0 : \dots : x_n]$ of \mathbf{P}_K^n such that $|x_i| = \max(|x_0|, \dots, |x_n|)$.

To fix ideas, let us consider $i = 0$. Then, $\mathcal{U}_0 = \text{Spec}(K^\circ[T_1, \dots, T_n])$ is the affine space over K° with coordinates $T_j = x_j/x_0$. The natural topology on the algebra $K^\circ[T_1, \dots, T_n]$, and on its tensor product with K , $K[T_1, \dots, T_n]$, is induced by the *Gauß norm*

$$\|f\| = \max_{\mathbf{a} \in \mathbb{N}^n} |f_{\mathbf{a}}|, \quad f = \sum f_{\mathbf{a}} T_1^{a_1} \dots T_n^{a_n}.$$

The completion of $K[T_1, \dots, T_n]$ for this norm is the Tate algebra and is denoted $K\langle T_1, \dots, T_n \rangle$. It consists of all power series $f = \sum f_{\mathbf{a}} T_1^{a_1} \dots T_n^{a_n}$ with coefficients in K such that $|f_{\mathbf{a}}| \rightarrow 0$ when $|\mathbf{a}| = a_1 + \dots + a_n \rightarrow \infty$; it is endowed with the natural extension of the Gauß norm, and is complete. By definition, the generic fiber U_0 of \mathcal{U}_0 in the sense of analytic geometry is the Berkovich spectrum of the Tate algebra, that is the set of all multiplicative semi-norms on it which are continuous with respect to the topology defined by the Gauß norm. Since the theorem of Gauß asserts that this norm is multiplicative, it defines a point $\gamma \in U_0$, which we like to call the *Gauß point*.

The reduction map $U_0 \rightarrow \mathcal{U}_0 \otimes \tilde{K}$ is defined as follows. Let $x \in U_0$, let $\mathfrak{p}_x \subset K\langle T_1, \dots, T_d \rangle$ be the kernel of the semi-norm x , which is also the kernel of the canonical morphism $\theta_x: K\langle T_1, \dots, T_d \rangle \rightarrow \mathcal{H}(x)$. The images $T_j(x)$ of the indeterminates T_j are elements of absolute value ≤ 1 of the complete ultrametric field $\mathcal{H}(x)$; they belong to its valuation ring $\mathcal{H}(x)^\circ$. Letting $\overline{\mathcal{H}(x)}$ to be the residue field, there exists a unique morphism $\theta_{\bar{x}}: \tilde{K}[T_1, \dots, T_d] \rightarrow \overline{\mathcal{H}(x)}$ such that $\theta_{\bar{x}}(T_i)$ is the image in $\overline{\mathcal{H}(x)}$ of $T_i(x)$. The kernel of this morphism is a prime ideal of the ring $\tilde{K}[T_1, \dots, T_d]$ and defines a point \bar{x} in the scheme $\mathcal{U}_0 \otimes \tilde{K}$.

Let us now compute the reduction of the Gauß point γ . By definition, the field $\mathcal{H}(\gamma)$ is the completion of the Tate algebra $K\langle T_1, \dots, T_d \rangle$ for the Gauß norm. I claim that morphism $\theta_{\bar{\gamma}}$ is injective, in other words, that the images of $T_1(\gamma), \dots, T_d(\gamma)$ in the residue field $\overline{\mathcal{H}(\gamma)}$ are algebraically independent. Let $P \in K^\circ[T_1, \dots, T_d]$ be any polynomial whose reduction \bar{P} belongs to the kernel of $\theta_{\bar{\gamma}}$; this means $|P|_\gamma < 1$; in other words, the Gauß norm of P is < 1 and each coefficient of P has absolute value < 1 . Consequently, $\bar{P} = 0$ and $\theta_{\bar{\gamma}}$ is injective, as claimed. This shows that $\bar{\gamma}$ is the generic point of the scheme $\mathcal{U}_0 \otimes \tilde{K}$.

We thus have proved the following proposition.

2.1.1. Proposition. — *The measure $c_1(\overline{\mathcal{O}(1)}_W)^n$ on \mathbf{P}_K^n is the Dirac measure at the Gauß point γ .*

2.2. Semi-stable curves and reduction graphs

In this section, we assume that X is the analytic space associated to a projective curve over a field K which is complete for a discrete valuation. The semi-stable reduction theorem of Deligne–Mumford asserts that, up to replacing the base field K by a finite extension, the curve X has a projective model \mathfrak{X} over K° which is regular (as a 2-dimensional scheme) and whose special fiber is reduced, with at most double points for singularities. We may also assume that the irreducible components are geometrically irreducible. We do not require, however, that \mathfrak{X} is the minimal semi-stable model.

2.2.1. THE REDUCTION GRAPH OF THE SPECIAL FIBER. — In that situation, the *reduction graph* $R(\mathfrak{X})$ is a metrized graph defined as follows. It has for vertices the irreducible components of the special fiber, with as many edges of length $\log|\pi|^{-1}$ between two vertices as the number of intersection points of the corresponding components. In a neighborhood of a double point, \mathfrak{X} looks like (*i.e.*, has an étale map to) the scheme with equation $xy = \pi$ in the affine plane $\mathbf{A}_{K^\circ}^2$.

If one replaces the field K by a finite extension K' , the base change $\mathfrak{X} \otimes_{K^\circ} (K')^\circ$ may no longer be regular. Indeed, $\mathfrak{X} \otimes_{K^\circ} (K')^\circ$ is étale locally isomorphic to $xy = (\pi')^e$, where π' is a uniformizing element of K' , and e is the ramification index. When $e > 1$, the origin is a singular point of that scheme and one needs to blow it up repeatedly in order to obtain a regular scheme, which is a semi-stable model of $X_{K'}$ over $(K')^\circ$. The two initial components are replaced by a chain of $e + 1$ components, the $e - 1$ intermediate ones being projective lines. In other words, $e - 1$ vertices have been added, regularly spaced along each edge. One concludes that the reduction graph has not changed, as a topological space. Its metric has not changed either, since the e edges that partition an original edge (of length $\log|\pi|^{-1}$) have length $\log|\pi'|^{-1} = \frac{1}{e} \log|\pi|^{-1}$.

We say that a function on $R(\mathfrak{X})$ is piecewise linear if, up to passing to a finite extension (which replaces each edge by e edges of length equal to $1/eth$ of the initial one), it is linear on each edge.

2.2.2. DRAWING THE REDUCTION GRAPH ON THE BERKOVICH SPACE. — Let us analyse the situation from the Berkovich viewpoint. As we have seen, the generic points of the special fiber are the reductions of canonical points of X : the vertices of the graph $R(\mathfrak{X})$ naturally live in X . The same holds for the edges, but is a bit more subtle. As we have seen, blowing-up intersection points of components in the special fiber gives rise to new components, hence to new points of X . Would we enlarge the ground field and blow-up indefinitely, the constellation of points in X that we draw converges to a graph which is isomorphic to $R(\mathfrak{X})$.

According to Berkovich [13], a far more precise result holds. Let us consider a neighborhood \mathcal{U} of a singular point of the special fiber, pretending it is isomorphic to the locus defined by the equation $xy = \pi$ in \mathbf{A}^2 ; so $\mathcal{U} = \text{Spec}(K^\circ[x, y]/(xy - \pi))$. Its generic fibre is the affinoid space U defined by the equality $|xy| = |\pi|$ in the unit polydisk $B^2 = \mathcal{M}(K\langle x, y \rangle)$. The affinoid algebra of U is the quotient

$$K\langle x, y \rangle / (xy - \pi)$$

whose elements f are (non-uniquely) represented by a series

$$\sum_{m,n=0}^{\infty} a_{m,n} x^m y^n,$$

with $a_{m,n} \rightarrow 0$ when $m+n \rightarrow \infty$. However, observing that x is invertible in this algebra, with inverse $\pi^{-1}y$, so that $y = \pi x^{-1}$, we can replace each product $x^m y^n$ by $\pi^n x^{m-n}$, leading to an expression of the form

$$f = \sum_{n \in \mathbf{Z}} a_n x^n,$$

where $|a_n| \rightarrow 0$ when $n \rightarrow +\infty$ and $|a_n| \pi^{-n} \rightarrow 0$ when $n \rightarrow -\infty$. Such an expression is now unique, and is called the Laurent expansion of f .

It leads to a natural family $(\gamma_r)_{r \in [0, \log|\pi|^{-1}]}$ of multiplicative seminorms on the algebra $\mathcal{O}(U)$, parametrized by the unit interval in \mathbf{R} . Namely, for each real number $r \in [0, \log|\pi|^{-1}]$, we can set

$$\gamma_r(f) = \max_{n \in \mathbf{Z}} |a_n| e^{-rn}, \quad f = \sum_{n \in \mathbf{Z}} a_n x^n \in \mathcal{O}(U).$$

Obviously, γ_r is a norm on $\mathcal{O}(U)$ which extends the absolute value of K ; its multiplicativity is proved analogously to that of the Gauß norm. It is easy to check that the map $[0, \log|\pi|^{-1}] \rightarrow U$ defined by $r \mapsto \gamma_r$ is continuous (this amounts to the fact that the maps $r \mapsto \gamma_r(f)$ are continuous), hence defines an parametrized path in the topological space U .

Let $S(\mathfrak{U})$ be its image (with the induced distance); Berkovich calls it the skeleton of the formal scheme obtained by completing \mathfrak{U} along its special fibre. A point u in U has two coordinates $(x(u), y(u))$ in the completed residue field $\mathcal{H}(u)$ which are elements of absolute value ≤ 1 satisfying $x(u)y(u) = \pi$. In particular,

$$r(u) = \log|x(u)|^{-1} \in [0, \log|\pi|^{-1}].$$

The map $\rho: u \mapsto \gamma_{r(u)}$ is a continuous function from U to $S(\mathfrak{U})$.

Let us compute the image of γ_r by this map. By definition of γ_r , one has

$$|x(\gamma_r)| = \gamma_r(x) = e^{-r},$$

hence $r(\gamma_r) = r$ and $\rho(\gamma_r) = \gamma_r$. In other words, the map ρ is a retraction of U onto the skeleton $S(\mathfrak{U})$.

The special fiber of \mathfrak{U} is defined by the equation $xy = 0$ in $\mathbf{A}_{\bar{K}}^2$, hence has two components. One can check that the point γ_0 reduces to the generic point of the component with equation $y = 0$, while $\gamma_{\log|\pi|^{-1}}$ reduces to the generic point of the component with equation $x = 0$.

These constructions have to be done around each singular point of the special fiber of \mathfrak{X} , locally for the étale topology of \mathfrak{X} . Berkovich proves that they can be glued, so that the graph $R(\mathfrak{X})$ is again canonically interpreted as an actual metrized graph drawn on the analytic space X ; we write $\iota: R(\mathfrak{X}) \hookrightarrow X$ for the canonical embedding. The map ι admits a continuous retraction $\rho: X \rightarrow R(\mathfrak{X})$.

Although we will not use this fact, we must mention that the retraction ρ is a deformation retraction.

2.2.3. METRIZED LINE BUNDLES AND THE REDUCTION GRAPH. — A construction of S. Zhang [61], building on prior results of Chinburg–Rumely [22], furnishes continuous metrics on divisors from continuous functions on the reduction graph $R(\mathfrak{X})$. It works as follows. First of all, if $P \in X(K)$ is a rational point, there is a unique morphism $\varepsilon_P: \text{Spec } K^\circ \rightarrow \mathfrak{X}$ which extends the point P viewed as a morphism from $\text{Spec } K$ to X . The image of this section is a divisor D_P on \mathfrak{X} and the line bundle $\mathcal{O}(D_P)$ on \mathfrak{X} defines a smooth metric on $\mathcal{O}(P)$; we write $\overline{\mathcal{O}(P)}_{\mathfrak{X}}$ for the corresponding metrized line bundle. We also define μ_P as the Dirac measure at the vertex of the graph corresponding to the (unique) irreducible component of the special fiber by which D_P passes through. The construction and the notation is extended by additivity for divisors which are sums of rational points. More generally, if P is only a closed point of X , we do this construction after the finite extension $K(P)/K$, so that P becomes a sum of rational points, using for model the minimal resolution of $\mathfrak{X} \otimes K(P)^\circ$ described earlier.

If f is any continuous function on $R(\mathfrak{X})$ and D a divisor on X , the metrized line bundle $\mathcal{O}(D + f)_{\mathfrak{X}}$ is deduced from $\overline{\mathcal{O}(D)}_{\mathfrak{X}}$ by multiplying the metric by e^{-f} . When f is piecewise linear, this metrized line bundle is smooth. To prove that, we may extend the scalars and assume that D is a sum of rational points $\sum n_j P_j$ and that f is linear on each edge corresponding to an intersection point of components of the special fiber. Letting (V_i) being the family of these components, and writing v_i for the vertex of $R(\mathfrak{X})$ corresponding to V_i , the divisor

$$(2.2.4) \quad \sum_j n_j D_{P_j} + \sum_i f(v_i) V_i$$

defines the metrized line bundle $\mathcal{O}(D + f)_{\mathfrak{X}}$.

In this context, Zhang has defined a curvature operator, which associates to a metrized line bundle a distribution on the graph $R(\mathfrak{X})$, defined in such a way that

- for any divisor D on X , $\text{curv}(\overline{\mathcal{O}(D)}_{\mathfrak{X}}) = \mu_D$;
- for any continuous function f , $\text{curv}(\overline{\mathcal{O}(f)}_{\mathfrak{X}}) = -\Delta f$, where Δ is the Laplacian operator of the graph $R(\mathfrak{X})$,

and depending linearly on the metrized line bundle. The following lemma compares this construction with the general one on Berkovich spaces.

2.2.5. Lemma. — *Let $\overline{L} = \overline{\mathcal{O}(D + f)}_{\mathfrak{X}}$ be a metrized line bundle on X associated to a divisor D on X and a continuous function f on the graph $R(\mathfrak{X})$. If it is semi-positive, resp. admissible in the sense of [61], then it is semi-positive, resp. admissible in the sense of this article, and one has*

$$c_1(\overline{L}) = \iota_* \text{curv}(\overline{\mathcal{O}(D + f)}) \log |\pi|^{-1}.$$

In other words, the measure $c_1(\overline{L})$ is supported by the graph $R(\mathfrak{X})$ where it coincides essentially with Zhang’s curvature.

Proof. — We first assume that f is linear on each edge of $R(\mathfrak{X})$ and that D is a sum of rational points of X . Then, \overline{L} corresponds to the line bundle \mathcal{L} on the model \mathfrak{X} given by

Equation 2.2.4. By definition, the measure $c_1(\mathfrak{L})$ is computed as follows. It is a sum, for all components V_i of the special fiber, of $\deg(\mathfrak{L}|V_i) \log|\pi|^{-1}$ times the Dirac measure at the corresponding point v_i of $R(\mathfrak{X})$. In particular, it is supported by $R(\mathfrak{X})$. Then,

$$\deg(\mathfrak{L}|V_i) = \sum_j n_j \left\{ \begin{array}{ll} 1 & \text{if } D_{P_j} \text{ passes through } V_i; \\ 0 & \text{otherwise} \end{array} \right\} + \sum_j f(V_j)(V_i, V_j),$$

where (V_i, V_j) is the intersection number of the divisors V_i and V_j . That D_{P_j} passes through V_i means exactly that $\rho(P_j) = v_i$. Moreover, if $j \neq i$, then $(V_i, V_j) = m_{i,j}$ is just the number of intersection points of V_i and V_j , while

$$(V_i, V_i) = (V_i, \sum_j V_j) - \sum_{j \neq i} (V_i, V_j) = - \sum_{j \neq i} (V_i, V_j),$$

since the whole special fiber is numerically equivalent to zero. Consequently,

$$\sum_j f(v_j)(V_i, V_j) = \sum_{j \neq i} m_{i,j} (f(V_j) - f(V_i)).$$

Observe that this is the sum, over all edges from V_i , of the derivative of f along this edge. Comparing with the definitions given by Zhang in [61], one finds, for any function g on $R(\mathfrak{X})$

$$\begin{aligned} \sum_i \deg(\mathfrak{L}|V_i) g(v_i) &= \sum_j n_j g(\rho(P_j)) + \sum_i \langle \delta f(v_i), g \rangle \\ &= \int_{R(\mathfrak{X})} g (\mu_D + \delta f) \\ &= \int_{R(\mathfrak{X})} g \operatorname{curv}(\overline{\mathcal{O}(D + f)}). \end{aligned}$$

This proves the claimed formula when f is linear on each edge of \mathfrak{X} and D is a sum of rational points.

By working over an appropriate finite extension of K , it extends to the case where f is only piecewise linear, D being any divisor on X .

Zhang defines $\overline{\mathcal{O}(D + f)}_{\mathfrak{X}}$ to be semi-positive if f is a uniform limit of piecewise linear functions f_n such that $\operatorname{curv}(\overline{\mathcal{O}(D + f_n)}_{\mathfrak{X}}) \geq 0$. The metrized line bundle \overline{L} is then the limit of the metrized line bundles \overline{L}_n corresponding to models \mathfrak{L}_n (on appropriate models \mathfrak{X}_n of X after some extension of scalars) of $\mathcal{O}(D)$. By the previous computation, these metrics are smooth and $c_1(\overline{L}_n) \geq 0$. Reversing the computation, this means that \mathfrak{L}_n is numerically effective on \mathfrak{X}_n , hence \overline{L} is semi-positive. By definition of the measure $c_1(\overline{L})$, one has

$$\begin{aligned} c_1(\overline{L}) &= \lim_n c_1(\overline{L}_n) = \lim_n \iota_* \operatorname{curv}(\overline{\mathcal{O}(D + f_n)}) \\ &= \iota_* \lim_n \operatorname{curv}(\overline{\mathcal{O}(D + f_n)}) = \iota_* \operatorname{curv}(\overline{\mathcal{O}(D + f)}). \end{aligned}$$

The case of an admissible metrized line bundle follows by linearity. \square

2.3. Local character of the measures

The definition of the measures associated to metrized line bundles is global in nature. Still, the main result of this section implies that they are local.

2.3.1. Definition. — Let X be an analytic space. A function on X is said to be *strongly pluriharmonic* if it is locally a uniform limit of functions of the form $a \log |u|$, where $a \in \mathbf{R}$ and u is holomorphic and nonvanishing.

There is a general theory of harmonic functions on curves due to Thuillier [55] (see also [33, 8] on the projective line; note that the definition of a strongly harmonic function of the latter reference is different from the one adopted here). Strongly pluriharmonic functions are harmonic in their sense. Indeed, logarithms of absolute values of invertible holomorphic functions are harmonic, and harmonic functions are preserved by uniform limits (Prop. 2.3.20 and 3.1.2 of [55]). In fact, when the residue field of K is algebraic over a finite field, any harmonic function is locally of the form $a \log |u|$, where $a \in \mathbf{R}$ and u is an invertible holomorphic function (*loc.cit.*, Theorem 2.3.21).

This is not necessarily the case for more general fields K : there are harmonic functions over analytic curves which are not locally equal to the logarithm of the absolute value of an invertible function; examples require to consider curves of genus ≥ 1 . In a conversation with A. Ducros, we devised the following example of a one-dimensional affinoid space. Let \mathcal{E} be an elliptic scheme over K° , let o be the origin in $\mathcal{E}_{\bar{K}}$ and let p be a *non-torsion* rational point in $\mathcal{E}_{\bar{K}}$; let \mathcal{X} be the blow-up of \mathcal{E} at the point p . Let then \mathcal{U} be its open subset obtained by removing the point o as well as a smooth point in the exceptional divisor of the blow-up; its generic fiber U is the desired affinoid space — it is the complementary subset in the elliptic curve E_K to two small disjoint disks. One can prove that the space of harmonic functions on U is 2-dimensional, and that all holomorphic invertible functions on U have constant absolute value.

I do not know whether any harmonic function on a curve is *locally* a uniform limit of logarithms.

2.3.2. Definition. — Let \bar{L} be a metrized line bundle on an analytic space X and let U be an open subset of X . One says that \bar{L} is *strongly pluriharmonic* on U if for any local frame s of L defined on an open subset $V \subset U$, $\log \|s\|^{-1}$ is strongly pluriharmonic on V .

A metrized line bundle is strongly pluriharmonic on U if it admits, in a neighbourhood of any point of U , a local frame whose norm is identically equal to 1. (For the converse to hold, one would need to introduce a notion of local frame for real line bundles.)

2.3.3. Proposition. — Let X be the analytic space associated to a proper K -scheme. Let $\bar{L}_1, \bar{L}_2, \dots, \bar{L}_k$ be admissible metrized line bundles on X . Let Z be a k -dimensional Zariski closed subset of X . Assume that \bar{L}_1 is strongly pluriharmonic on U . Then, the support of the measure $c_1(\bar{L}_1) \dots c_1(\bar{L}_k) \delta_Z$ is disjoint from U .

Proof. — One has to show that for any continuous function φ with compact support contained in U

$$\int_X \varphi c_1(\bar{L}_1) \dots c_1(\bar{L}_k) \delta_Z = 0.$$

By Gubler's theorem, the space of smooth functions is dense in the space of continuous functions on X . Using the fact that the maximum and the minimum of smooth functions are still smooth, one proves that the space of smooth functions with compact support contained in U is dense in the space of continuous functions with compact support contained in U , for the topology of uniform convergence. We thus may assume that φ is smooth, with compact support contained in U . Finally, we may also assume that the metric on the line bundles $\bar{L}_2, \dots, \bar{L}_k$ are smooth.

We may argue locally and assume that L_1 has a meromorphic section s whose divisor $\text{div}(s)$ is disjoint from U . Up to shrinking U again, we may assume that there exists a sequence (u_n) of rational functions without zeroes nor poles on U such that $\log \|s\| = \lim \log |u_n|^{1/n}$.

According to Prop. 1.3.14, one has

$$\begin{aligned} \int_X \varphi c_1(\bar{L}_1) \dots c_1(\bar{L}_k) \delta_Z \\ = \int_X \varphi c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_{\text{div}(s|_Z)} + \int_X \log \|s\|^{-1} dd^c \varphi c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_Z, \end{aligned}$$

The first term vanishes because $\text{div}(s|_Z)$ and the support of φ are disjoint. The second is the limit of

$$\int_X \log |u_n|^{-1/n} dd^c \varphi c_1(\bar{L})^{k-1} \delta_Z.$$

Using the fact that $\text{div}(u_n) \cap U$ is empty and applying the same computation, the term of index n equals

$$\frac{1}{n} \int_X \varphi c_1(\bar{M}_n) c_1(\bar{L}_2) \dots c_1(\bar{L}_k) \delta_Z.$$

where \bar{M}_n is the trivial metrized line bundle \mathcal{O}_X , and its meromorphic section u_n replacing s . But this integral is zero, by the formula (1.3.9) which defines the measure associated to smooth metrized line bundles. \square

2.4. Polarized dynamical systems

We now explain another example of metrized line bundles: the canonical metric associated to a dynamical system.

2.4.1. Lemma. — *Let X be the analytic space associated to a proper K -scheme and let $f: X \rightarrow X$ be a finite morphism. Let L be a line bundle on X , d an integer such that $d \geq 2$ and an isomorphism $\varepsilon: f^*L \simeq L^d$. The line bundle L possesses a unique continuous metric such that the isomorphism ε is an isometry. If L is ample, then this metric is semi-positive.*

In essence, this result, or at least its proof, goes back to Tate's construction of the "Néron–Tate" canonical height for abelian varieties. In the slightly different language

of local heights and Néron functions, it has been proved by Call–Silverman [17]. In the asserted form, it is due to Zhang [63].

Proof. — Let us first prove uniqueness. If \bar{L} and \bar{L}' are two metrics on L , let φ be the continuous function such that $\|\cdot\|' = e^{-\varphi} \|\cdot\|$. Assuming that ε is an isometry for these two metrics, one obtains the following equation

$$\varphi(f(x)) = d\varphi(x),$$

for any $x \in X$. Since X is compact, φ is bounded and this equation implies that $\|\varphi\|_\infty \leq \frac{1}{d} \|\varphi\|_\infty$. Since $d \geq 2$, one concludes that $\varphi \equiv 0$.

For the existence, one begins with any continuous metric \bar{L}_0 on L . Let us then consider the sequence of metrics (\bar{L}_n) on L induced by the pull-backs on $L^d = \varepsilon f^* L$, $L^{d^2} = (\varepsilon f^*)^2 L$, etc., hence on L . Since $d \geq 2$, a similar contraction argument as the one used for uniqueness shows that this is a Cauchy sequence of metrics on L ; consequently, it converges to a continuous metric on L . If \bar{L}_0 is chosen to be semi-positive, which we may if L is ample, then all of the metrized line bundles \bar{L}_n are semi-positive, hence the canonical metric is semi-positive.

Concretely, in the non-archimedean case, one begins with a model $(\mathfrak{X}_0, \mathfrak{L}_0, e)$ such that \mathfrak{L}_0 is numerically effective. Then one considers the map $f: X \rightarrow \mathfrak{X}_0$ and the normalization \mathfrak{X}_1 of \mathfrak{X}_0 in X ; this is a projective model \mathfrak{X}_1 , equipped with a finite morphism $f_1: \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$ extending f . Moreover, $\mathfrak{L}_1 = f_1^* \mathfrak{L}_0$ is a model of $f^* L^e$ which is identified with L^{ed} via the fixed isomorphism ε . Iterating this construction defines a sequence $(\mathfrak{X}_n, \mathfrak{L}_n, ed^n)$ of models of (X, L) , with finite morphisms $f_n: \mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1}$ such that $f_n^* \mathfrak{L}_{n-1} = \mathfrak{L}_n$. The metric on L defined by any of these models is semi-positive, hence so is their uniform limit. \square

2.4.2. THE CANONICAL MEASURE. — The measure $c_1(\bar{L})^n$ on X defined by the metrized line bundle \bar{L} is a very important invariant of the dynamical system. It satisfies the functional equations

$$f^* c_1(\bar{L})^n = d^n c_1(\bar{L})^n \quad \text{and} \quad f_* c_1(\bar{L})^n = c_1(\bar{L})^n.$$

The first follows by a general functorial property proved in [19]; it implies the second. The support of the canonical measure is therefore totally invariant under f .

2.4.3. THE FATOU SET. — Generalizing results of Kawaguchi–Silverman in [44] and Baker–Rumely [8], we want to show here that the canonical measure vanishes on any open set U of X where the sequence $(f^n|_U)$ of iterates of f is equicontinuous.

Let U be an open set in X and \mathcal{F} be a family of continuous maps from U to X . One says that this family is equicontinuous if for any $x \in U$ and any finite open covering (V_j) of X , there exists a neighbourhood U_x of x in U such that for any $\varphi \in \mathcal{F}$, there exists an index j such that $\varphi(U_x) \subset V_j$. (This definition is adapted from Definition 10.63 in [8]; it is the definition of equicontinuity associated to the canonical uniform structure of the compact space X .)

We define the equicontinuous locus of f as the largest open subset E_f of X over which the sequence of iterates of f is equicontinuous.

2.4.4. Proposition. — *If L is ample, then the metric \bar{L} is strongly pluriharmonic on E_f .⁽⁴⁾*

Proof. — The proof is inspired from the above-mentioned sources, which in turns is an adaptation of the complex case [43] (see also [56]).

We may replace L by a positive power of itself and assume that it is very ample, induced by a closed embedding of X in P^n , and that the natural map $\Gamma(P^n, \mathcal{O}(d)) \rightarrow \Gamma(X, \mathcal{O}(d))$ is surjective. Then, there are homogeneous polynomials (F_0, \dots, F_n) , of degree d , with coefficients in K , and without common zeroes on X , such that $f([x_0 : \dots : x_n]) = [F_0(x) : \dots : F_n(x)]$ for any $x = [x_0 : \dots : x_n] \in P^n$. One considers the polynomial map $F: A^{n+1} \rightarrow A^{n+1}$; it lifts a rational map on P^n which extends the morphism f .

For $(x_0, \dots, x_n) \in A^{n+1}$, define $\|x\| = \max(|x_0|, \dots, |x_n|)$. The Weil metric on $\mathcal{O}(1)$ is given by

$$\log \|s_P(x)\|^{-1} = \log |P(x)|^{-1} + \deg(P) \log \|x\|,$$

where P is an homogeneous polynomial, s_P the corresponding global section of $\mathcal{O}(\deg(P))$, and x is a point of A^{n+1} such that $P(x) \neq 0$. The restriction to X of this metric is a semi-positive metric $\|\cdot\|_0$ on L . The construction of the canonical metric on L introduces a sequence of semi-positive metrics $\|\cdot\|_n$ on L ; these metrics are given by the following explicit formula

$$\log \|s_P(x)\|_k^{-1} = \log |P(x)|^{-1} + \deg(P) d^{-k} \log \|F^{(k)}(x)\|,$$

where $F^{(k)}: A^{n+1} \rightarrow A^{n+1}$ is the k th iterate of F .

The convergence of this sequence is therefore equivalent to the convergence of the sequence $(d^{-k} \log \|F^{(k)}\|)_k$ towards a continuous fonction on the preimage of X under the projection map $A^{n+1} \setminus \{0\} \rightarrow P^n$. The limit is usually called the homogeneous Green function.

For $0 \leq i \leq n$, let V_i be the open set of points $x = [x_0 : \dots : x_n] \in P^n$ such that $|x_i| > \frac{1}{2} \|x\|$. They form an open covering of P^n ; their intersections with X form an open covering of X .

Fix $x \in E_f$ and let U be an open neighbourhood of x such that for any positive integer k , there exists $i \in \{0, \dots, n\}$ such that $f^k(U) \subset V_i$. For any i , let N_i be the set of integers k such that $f^k(U) \subset V_i$. Let us consider any index i such that N_i is infinite; to fix ideas, let us assume that $i = 0$. The canonical norm of a section s_P at a point $y \in U$

4. The ampleness assumption should not be necessary for the result to hold.

is given by

$$\begin{aligned}
\log \|s_P(y)\|^{-1} &= \log |P(y)|^{-1} + \deg(P) \lim_{\substack{k \rightarrow \infty \\ k \in N_0}} d^{-k} \log \|F^{(k)}(y)\| \\
&= \log |P(y)|^{-1} \\
&\quad + \deg(P) \lim_{\substack{k \rightarrow \infty \\ k \in N_0}} d^{-k} \left(\log |F_0^{(k)}(y)| \right. \\
&\quad \left. + \log \max_{0 \leq i \leq m} |F_i^{(k)}(y)/F_0^{(k)}(y)| \right).
\end{aligned}$$

Observe that $[F_0^{(k)}(y) : \dots : F_m^{(k)}(y)]$ are the homogeneous coordinates of the point $f^k(y)$. Since $y \in U$ and $f^k(U) \subset V_0$, one has $|F_i^{(k)}(y)| \leq 2 |F_0^{(k)}(y)|$, so that the last term is bounded by $d^{-k} \log 2$ and uniformly converges to 0 on U . Finally, uniformly on U ,

$$\log \|s_P(y)\|^{-1} = \log |P(y)|^{-1} + \deg(P) \lim_{\substack{k \rightarrow \infty \\ k \in N_0}} d^{-k} \log |F_0^{(k)}(y)|.$$

This shows that $\log \|s_P\|^{-1}$ is strongly harmonic on U , as claimed. \square

2.4.5. Corollary. — *The canonical measure $c_1(\bar{L})^n$ vanishes on E_f .*

Proof. — It suffices to apply Prop. 2.3.3. \square

2.4.6. REMARKS. — 1) The particular case $X = \mathbb{P}^n$ generalizes Theorem 6 in [44] according to which canonical metrics are locally constant on the classical Fatou set (meaning that the norm of a non-vanishing local section is locally constant). Indeed, the restriction to the set of smooth rigid points of a strongly harmonic function is locally constant. This follows from the fact that any such point has an affinoid neighbourhood U which is a polydisk, so that the absolute value of any invertible function on U , hence any harmonic function on U is constant.

2) In the case $X = \mathbb{P}^1$, Fatou and Julia sets in the Berkovich framework have been studied by Rivera-Letelier [52] and Benedetto [11]; see also [8] for a detailed exposition of the theory and further references. An example of Rivera-Letelier on the projective line (Example 10.70 of [8]) shows that the equicontinuity locus E_f may be smaller than the complement of the support of the measure $c_1(\bar{L})$.

Anyway, this proposition suggests the interest of a general study of Fatou sets and of pluripotential theory on Berkovich spaces. For example, is there an interesting theory of pseudoconvexity for Berkovich spaces? Is it related to Stein spaces? By analogy to the complex case (see [56]), are Berkovich Fatou components pseudoconvex? Stein?

2.5. Abelian varieties

Let us assume throughout this section that X is an Abelian variety. For any integer m , let $[m]$ be the multiplication-by- m endomorphism of X .

2.5.1. CANONICAL METRICS. — Let L be a line bundle on X . Let 0 be the neutral element of X and let us fix a trivialization L_0 of L at 0 .

The line bundle $L^{\otimes 2}$ is canonically decomposed as the tensor product of an even and an odd line bundle:

$$L^{\otimes 2} = (L \otimes [-1]^* L) \otimes (L \otimes [-1]^* L^{-1}).$$

By the theorem of the cube, an even line bundle L satisfies $[m]^* L \simeq L^{\otimes m^2}$, while for an odd line bundle L , one has $[m]^* L \simeq L^{\otimes m}$; moreover, there are in each case a unique isomorphism compatible with the trivialization at the origin. By Lemma 2.4.1, an even (resp. an odd) line bundle possesses a canonical continuous metric making this isomorphism an isometry. This furnishes a *canonical metric* on $L^{\otimes 2}$, hence on L . According to this lemma, this metric is semi-positive if L is ample and even. Using a Lemma of Künnemann, ([18], Lemme 2.3), one proves that this also holds if L is algebraically equivalent to 0 . In any case, the canonical metrics are admissible.

2.5.2. THE CASE OF GOOD REDUCTION. — When the variety X has good reduction, the canonical metrics and the associated measures are fairly easy to describe. Indeed, let \mathfrak{X} be the Néron model of X over K° , an Abelian scheme. For any line bundle L on X there is a unique line bundle \mathfrak{L} on \mathfrak{X} which extends L and which admits a trivialization at the 0 section extending the given one over K . By the theorem of the cube for the Abelian scheme \mathfrak{X} , the isomorphism $[m]^* L \simeq L^{\otimes m^a}$ (with $a = 1$ or 2 , according to whether L is odd or even) extends uniquely to an isomorphism $[m]^* \mathfrak{L} \simeq \mathfrak{L}^{\otimes m^a}$. This implies that the canonical metrics are smooth, induced by these models.

The description of the canonical measures on \mathfrak{X} follows at once. Let ξ be the point of X whose reduction is the generic point of the special fiber of \mathfrak{X} . Then, for any family (L_1, \dots, L_n) of line bundles on X , one has

$$c_1(\bar{L}_1) \dots c_1(\bar{L}_n) = \deg(c_1(L_1) \dots c_1(L_n)) \delta_\xi.$$

We see in particular that they only depend on the classes of the line bundles L_j modulo numerical equivalence.

2.5.3. GUBLER'S DESCRIPTION. — W. Gubler [41] has computed the canonical measures when the Abelian variety has bad reduction. We describe his result here.

Up to replacing K by a finite extension, we assume that X has split semi-stable reduction. Raynaud's uniformization involves an analytic group E which is an extension of an abelian variety with good reduction Y by a split torus $T \simeq \mathbf{G}_m^t$, where $t \in \{1, \dots, n\}$ — the so-called Raynaud extension of X . One has $t \geq 1$ since we assume *bad* reduction; moreover, $\dim Y = \dim E - t = n - t$. There is a morphism $p: E \rightarrow X$, whose kernel is a discrete subgroup M of $E(K)$, so that the induced map $E/\Lambda \rightarrow X$ is an isomorphism. When $t = n$, one says that X has totally degenerate reduction, and the morphism p is the rigid analytic uniformization of the abelian variety X .

Moreover, E is constructed as a contracted product $(E_1 \times T)/T_1$ from an extension E_1 of Y by the “unit subtorus” T_1 of T (defined by the equalities $|T_j(x)| = 1$ for $j \in \{1, \dots, t\}$ and $x \in T$). The natural map $\lambda_T: T \rightarrow \mathbf{R}^t$ defined by

$$x \mapsto (-\log |T_1(x)|, \dots, -\log |T_t(x)|)$$

is continuous and surjective; it admits a canonical section ι_T which maps a point $(u_1, \dots, u_t) \in \mathbf{R}^t$ to the semi-norm

$$f \mapsto \sup_{\mathbf{m} \in \mathbf{Z}^t} a_{\mathbf{m}} e^{-m_1 u_1 - \dots - m_t u_t}, \quad \text{for } f = \sum_{\mathbf{m}} a_{\mathbf{m}} T_1^{m_1} \dots T_t^{m_t} \in \mathcal{O}(T).$$

The map λ_T extends uniquely to a morphism $\lambda: E \rightarrow \mathbf{R}^t$ whose kernel contains E_1 . The image $\Lambda = \lambda(M)$ is a lattice of \mathbf{R}^t , and the morphism p induces a continuous proper morphism $\rho: X \rightarrow \mathbf{R}^t / \Lambda$. Composing the section ι_T with the projection p furnishes a section $\iota: \mathbf{R}^t / \Lambda \rightarrow X$ of ρ . Its image is the *skeleton* of X . Gubler's theorem ([41], Cor. 7.3) is the following:

2.5.4. Theorem. — *Let L_1, \dots, L_n be line bundles on X . The canonical measure $c_1(\bar{L}_1) \dots c_1(\bar{L}_n)$ is the direct image by ι of the unique Haar measure on \mathbf{R}^t / Λ whose total mass is $\deg(L_1 \dots L_n)$.*

3. Applications to Arakelov geometry

We now describe some applications of the previous considerations to arithmetic geometry over global fields.

3.1. Adelic metrics and heights

3.1.1. ADELIC METRICS. — Let F be either a number field (arithmetic case), or a finite extension of the field of rational functions over a constant field (geometric case). Let $M(F)$ be the set of normalized absolute values on F . Let X be a projective variety over F . Any $v \in M(F)$ gives rise to a complete valued field F_v , and to an analytic space X_v over F_v : if v is archimedean, $X_v = X(\bar{F}_v)$, while X_v is the Berkovich analytic space attached to X_{F_v} if v is ultrametric.

If L is a line bundle on X , an *adelic metric* on L is a family $(\|\cdot\|_v)_{v \in M(F)}$ of continuous metrics on the induced line bundles over the analytic spaces X_v . We require the following supplementary compatibility assumption: there exists a model $(\mathfrak{X}, \mathcal{L}, e)$ over the ring of integers of F inducing the given metrics at almost all places v . An adelic metric is said to be semi-positive, *resp.* admissible if it is so at all places of F .

Line bundles on X endowed with an adelic metric form a group $\overline{\text{Pic}}(X)$; admissible line bundles form a subgroup $\overline{\text{Pic}}_{\text{ad}}(X)$. If $f: Y \rightarrow X$ is any morphism, there is a natural morphism of groups $f^*: \overline{\text{Pic}}(X) \rightarrow \overline{\text{Pic}}(Y)$; it maps $\overline{\text{Pic}}_{\text{ad}}(X)$ into $\overline{\text{Pic}}_{\text{ad}}(Y)$.

3.1.2. HEIGHTS. — Consider line bundles $\bar{L}_0, \dots, \bar{L}_n$ with admissible adelic metrics. Let Z be a subvariety of X of dimension k and s_0, \dots, s_k invertible meromorphic sections of L_0, \dots, L_k whose divisors have no common intersection point on Z . For any $v \in M(F)$, we have recalled in Sections 1.2.5, 1.2.10 and 1.3.11 the definitions of the local height pairing

$$(\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k) | Z)_v$$

where the index ν indicates the corresponding place of F . The global height is the sum, over all $\nu \in M(F)$, of these local heights:

$$(\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z) = \sum_{\nu \in M(F)} (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z)_\nu.$$

It inherits from the local heights their multilinear symmetric character.

Let us replace s_k by another invertible meromorphic section $f s_k$. Then,

$$\begin{aligned} (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(f s_k)|Z) &= \sum_{\nu \in M(F)} (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(f s_k)|Z)_\nu \\ &= \sum_{\nu \in M(F)} (\widehat{\text{div}}(s_0) \dots \widehat{\text{div}}(s_k)|Z)_\nu \\ &\quad + \sum_{\nu \in M(F)} \int_{X_\nu} \log |f|^{-1} c_1(\bar{L}_0) \dots c_1(\bar{L}_{k-1}) \delta_{Z_\nu}. \end{aligned}$$

In particular, if Z is a point $z \in X(F)$, then $\delta_{Z_\nu} = \delta_z$ is the Dirac mass at z and

$$(\widehat{\text{div}}(s_0)|Z) = \sum_{\nu \in M(F)} \log \|s_0\|_\nu^{-1}(z).$$

Let us observe that it is independent on the choice of the chosen meromorphic section s_0 , provided it is regular at z . Any other section has the form $f s_0$, for some invertible meromorphic function f on X . Then,

$$\begin{aligned} (\widehat{\text{div}}(f s_0)|Z) &= \sum_{\nu \in M(F)} \log \|f s_0\|_\nu^{-1}(z) \\ &= \sum_{\nu \in M(F)} \log \|s_0\|_\nu^{-1}(z) + \sum_{\nu \in M(F)} \log |f|_\nu^{-1}(z) \\ &= (\widehat{\text{div}}(f s_0)|Z) \end{aligned}$$

since, by the product formula, the second term vanishes.

By induction on the dimension of Z , and using the commutativity of the local height pairings, it follows that the global height only depends on the metrized line bundles, and not on the actual chosen sections s_0, \dots, s_k . We denote it by

$$(\widehat{c}_1(\bar{L}_0) \dots \widehat{c}_1(\bar{L}_k)|Z).$$

Again, it is multilinear symmetric in the metrized line bundles $\bar{L}_0, \dots, \bar{L}_k$. By the same argument, it only depends on their isomorphism classes in $\overline{\text{Pic}}_{\text{ad}}(X)$.

It satisfies a projection formula: for any morphism $f: Y \rightarrow X$ and any k -dimensional subvariety Z of Y ,

$$(\widehat{c}_1(f^* \bar{L}_0) \dots \widehat{c}_1(f^* \bar{L}_k)|Z) = (\widehat{c}_1(\bar{L}_0) \dots \widehat{c}_1(\bar{L}_k)|f_*(Z)),$$

where the cycle $f_*(Z)$ is defined as $\deg(Z/f(Z))f(Z)$ if Z and $f(Z)$ have the same dimension, so that $f: Z \rightarrow f(Z)$ is generically finite, of some degree $\deg(Z/f(Z))$. If Z and $f(Z)$ don't have the same dimension, one sets $f_*(Z) = 0$.

3.1.3. HEIGHTS OF POINTS. — The height of an algebraic point is an important tool in Diophantine geometry. If \bar{L} is a line bundle with an adelic metric on X , then for any point $P \in X(F)$, viewed as a closed subscheme of X , one has

$$h_{\bar{L}}(P) = (\hat{c}_1(\bar{L})|P) = \sum_v \log \|s\|_v^{-1}(P),$$

where s is any meromorphic section on L which has neither a zero nor a pole at P . More generally, let $P \in X(\bar{F})$ be an algebraic point and let $[P]$ be the corresponding closed point of X . Then,

$$h_{\bar{L}}(P) = \frac{1}{[F(P) : F]} (\hat{c}_1(\bar{L})|[P])$$

is the height of P with respect to the metrized line bundle \bar{L} . In fact, restricted to points, these definitions apply to any, not necessary admissible, continuous metric on L . The resulting function is a representative of the classical height function relative to L (which is only defined up to the addition of a bounded function).

Observe also the following functorial property of the height: If $f: Y \rightarrow X$ is a morphism and $P \in Y(\bar{F})$, then $h_{f^*\bar{L}}(P) = h_{\bar{L}}(f(P))$. Finally, recall that if F is a global field, then the height with respect to a metrized ample line bundle \bar{L} satisfies Northcott's finiteness property: for any integers d and B , there are only finitely many points $P \in X(\bar{F})$ such that $[F(P) : F] \leq d$ and $h_{\bar{L}}(P) \leq B$.

3.1.4. ZHANG'S INEQUALITY. — The essential minimum of the height $h_{\bar{L}}$ is defined as

$$e(\bar{L}) = \sup_{\emptyset \neq U \subset X} \inf_{P \in U(\bar{F})} h_{\bar{L}}(P),$$

where the supremum runs over non-empty open subsets of X . If L is big, then $e(\bar{L})$ is a real number. Another way to state its definition is the following: for any real number B , then the set

$$\{P \in X(\bar{F}) ; h_{\bar{L}}(P) \leq B\}$$

is Zariski dense if $B > e(\bar{L})$, and is not Zariski dense if $B < e(\bar{L})$.

Assume that \bar{L} is an ample line bundle on X , equipped with a semi-positive adelic metric. The (geometric/arithmetical) Hilbert-Samuel theorem implies the following inequality

$$e(\bar{L}) \geq \frac{(\hat{c}_1(\bar{L})^{n+1}|X)}{(n+1)(c_1(L)^n|X)}.$$

(See Zhang [63], as well as [40, 30] for more details in the geometric case). When X is a curve and F is a number field, Autissier [3] proved that the inequality holds for any ample line bundle with an admissible adelic metric (see [19]); this extends to the geometric case.

3.2. Mahler measures and heights of divisors

In this section, we assume that X is a projective geometrically integral smooth curve of positive genus g over F . For any place $v \in M(F)$, let X_v be the corresponding analytic curve.

3.2.1. — Let f be an invertible meromorphic function on X . Let us view it as an invertible meromorphic section of the trivial metrized line bundle $\overline{\mathcal{O}_X}$. Let \overline{L} be any line bundle on X with an admissible adelic metric. Then,

$$(\hat{c}_1(\overline{L})\hat{c}_1(\overline{\mathcal{O}_X})|X) = 0.$$

Moreover, according to Theorem 1.3 of [20] (see Section 1.3.13),

$$(\hat{c}_1(\overline{L})\hat{c}_1(\overline{\mathcal{O}_X})|X) = (\hat{c}_1(\overline{L})|\operatorname{div}(f)) + \sum_{v \in M(F)} \int_{X_v} \log|f|_v^{-1} c_1(\overline{L})_v.$$

In other words, this furnishes an integral formula for the height (relative to \overline{L}) of any divisor which is rationally equivalent to 0:

$$(\hat{c}_1(\overline{L})|\operatorname{div}(f)) = \sum_{v \in M(F)} \int_{X_v} \log|f|_v c_1(\overline{L})_v.$$

3.2.2. NÉRON–TATE HEIGHTS. — We want to apply this formula to a specific metrized line bundle on X . The Jacobian J of X is an Abelian variety of dimension g . We also choose a divisor D of degree 1 on X and correspondingly fix an embedding ι of X into J . (For this, we may need to enlarge the ground field F .) Finally, we let Θ be the theta divisor of J , defined as the image of X^{g-1} by the map $(x_1, \dots, x_{g-1}) \mapsto \sum_{j=1}^{g-1} \iota(x_j)$.

As described above, the line bundle $\mathcal{O}_J(\Theta)$ admits a canonical metrization; this induces a metrization on its inverse image $L = \iota^* \mathcal{O}_J(\Theta)$ on X . The metrized line bundle $\overline{\mathcal{O}_J(\Theta)}$ gives rise to the (theta) Néron–Tate height on J . Consequently, decomposing $\operatorname{div}(f) = \sum n_P P$, we obtain

$$\begin{aligned} (\hat{c}_1(\overline{L})|\operatorname{div}(f)) &= \sum n_P [F(P) : F] \hat{h}_\Theta(\iota(P)) \\ &= \sum n_P [F(P) : F] \hat{h}_\Theta([P - D]), \end{aligned}$$

where D is the fixed divisor of degree 1 on X .

3.2.3. CANONICAL MEASURES. — Since L has degree g , the measure $c_1(\overline{L})_v$ on X_v has total mass g ; let us define a measure of total mass 1 on X_v by

$$\mu_v = \frac{1}{g} c_1(\overline{L})_v.$$

When v is archimedean, the measure μ_v is the Arakelov measure on the Riemann surface $X_v(\mathbf{C})$. Let us recall its definition. Consider an orthonormal basis $(\omega_1, \dots, \omega_g)$ of $H^0(X, \Omega_X^1)$, i.e., a basis satisfying the relations

$$\int_{X_v(\mathbf{C})} \omega_j \wedge \overline{\omega_k} = \delta_{j,k} = \begin{cases} 1 & \text{if } j = k; \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\mu_v = \frac{1}{g} \sum_{j=1}^g \omega_j \wedge \overline{\omega_j}.$$

Let us now assume that v is ultrametric. By a theorem of Heinz [42], the metric on the line bundle \overline{L} coincides with the canonical metric defined by Zhang [61] using the

reduction graph of the minimal regular model of X ; see also [6] for a related interpretation in the framework of tropical geometry. This allows in particular to compute the measure μ_v : the reader will find in [61, Lemma 3.7] a quite explicit formula for μ_v , involving the physical interpretation of the graph as an electric network. (Zhang's computation generalizes Theorem 2.11 of the prior paper [22] by Chinburg and Rumely, the normalization is slightly different; see also [7].)

3.2.4. SUPERELLIPTIC CURVES. — The formulas of this section combine to the following: if $\operatorname{div}(f) = \sum n_P P$ is a divisor of an invertible meromorphic function on X ,

$$\sum n_P \hat{h}_\Theta([P - D]) = \sum_{v \in M(F)} \int_{X_v} \log |f(x)|_v d\mu_v(x).$$

As pointed out by R. De Jong [24], the case of superelliptic curves is particularly interesting. Indeed, such curves are presented as a ramified μ_N -covering $x: X \rightarrow \mathbf{P}^1$ of the projective line, which is totally ramified over the point at infinity, given by an equation $y^N = a(x)$, where a is a polynomial of degree $m > N$, prime to N . One has $g = \frac{1}{2}(N-1)(m-1)$.

Let us take for the divisor D the single point O over the point at infinity. For each point P in $X(F)$, $x - x(P)$ is a rational function on X which has a single pole of order N at infinity, and which vanishes along the fiber $x^{-1}(x(P))$ of x . The group of automorphisms of X acts transitively on this fiber, and respects the metrics, so that all of these points have the same Néron-Tate height. This implies the following formula

$$\hat{h}_\Theta(P - O) = \frac{1}{N} \sum_{v \in M(F)} \int_{X_v} \log |x - x(P)|_v d\mu_v$$

of [24]. The elliptic Mahler measure, defined by [29, 28] as a Shnirelman integral is therefore a natural integral when viewed on Berkovich spaces.

3.3. An equidistribution theorem

3.3.1. BOGOMOLOV'S CONJECTURE. — Let X be a projective smooth curve of genus $g \geq 2$ and let \bar{L} be an ample line bundle on X with a canonical metric inducing the Néron-Tate height. When F is a number field, Bogomolov conjectured in [15] that $e(\bar{L}) > 0$; this conjecture has been shown by Ullmo [57]. Its generalization to a subvariety X of an Abelian variety A , L being an ample line bundle on A with a canonical metric, asserts that $e(X, \bar{L}) > 0$ when X is not the translate of an abelian subvariety by a torsion point; it has been shown by Zhang [64].

Since $h_{\bar{L}}(P) = 0$ for any algebraic point $P \in A(\bar{F})$ which is a torsion point, these theorems imply in turn a theorem of Raynaud [50, 51] (formerly, a conjecture of Manin and Mumford) that the torsion points lying in a subvariety X of an abelian variety are not Zariski dense in X , unless X is itself the translate of an abelian subvariety by a torsion point.

The analogues of Bogomolov's and Zhang's conjecture in the geometric case is still open in general; see [39, 23] and the references therein for partial results.

3.3.2. — The proofs by Ullmo and Zhang of Bogomolov's conjecture make a fundamental use of an equidistribution principle which had been discovered together with Szpiro [54]. Let us first introduce a terminology: say a sequence (or a net) of algebraic points in a variety X over a number field is *generic* if any strict subvariety of X contains at most finitely many terms of the sequence.

Let \bar{L} be a line bundle on X with a semi-positive adelic metric. The idea of the equidistribution principle is to consider a generic sequence (x_j) such that $h_{\bar{L}}(x_j) \rightarrow e(\bar{L})$, i.e., realizing the equality in Zhang's inequality, and to use this inequality further, as a variational principle. Let v be a place of F ; for any n , let $\delta(x_j)_v$ be the probability measure on X_v which gives any conjugate of x_j the same mass, $1/[F(x_j) : F]$. The equidistribution theorem states that for a generic sequence (x_j) , the sequence of measures $(\delta(x_j)_v)$ on X_v converges vaguely towards the measure $c_1(\bar{L})_v^n / (c_1(L)^n |X)$.

In these papers, the equidistribution property was only investigated at an archimedean place, but the introduction of the measures on Berkovich spaces was motivated by potential equidistribution theorems on those. In [19], I was able to prove general results on curves only. Indeed, unless X is a curve, I needed an ampleness assumption on the metrized line bundle \bar{L} in order to apply Zhang's inequality to slight variations of it. This requirement has been removed by a paper of Yuan [59] who could understand *arithmetic volumes* beyond the ample case. Yuan's proof is an arithmetic analogue of an inequality of Siu [53] which Faber [30] and Gubler [40] used to prove the geometric case of the equidistribution theorem.

In [20], we considered more general variations of the metrized line bundles. The discussion in that article was restricted to the arithmetic case but the arguments extend to the geometric case.

3.3.3. Theorem. — *Let X be a projective variety of dimension n over F . Let \bar{L} be an ample line bundle on X with a semi-positive adelic metric such that $e(\bar{L}) = (\hat{c}_1(\bar{L})^{n+1} |X) = 0$. Let (x_j) be a generic sequence of algebraic points in X such that $h_{\bar{L}}(x_j) \rightarrow 0$. Then, for any line bundle \bar{M} on X with an admissible adelic metric,*

$$\lim_{j \rightarrow \infty} h_{\bar{M}}(x_j) = \frac{(\hat{c}_1(\bar{L})^n \hat{c}_1(\bar{M}) |X)}{(c_1(L)^n |X)}.$$

The particular case stated above is equivalent to *loc.cit.*, Lemma 6.1, as one can see by multiplying the metric on \bar{L} by an adequate constant at some place of F . Taking for M the trivial line bundle \mathcal{O}_X , with an admissible metric, one recovers the equidistribution theorems of Yuan, Faber and Gubler.

3.4. Lower bounds for heights and the Hodge index theorem

In the final section, we use the Hodge index theorem in Arakelov geometry to establish positive lower bounds for heights on curves. The results are inspired by recent papers [5, 49], and the proofs are borrowed from [48]. After they were conceived, I received the preprint [58] which proves a similar result in any dimension.

3.4.1. THE ARITHMETIC HODGE INDEX THEOREM. — Let X be a projective smooth curve over F , let \bar{L} be a line bundle of degree 0 on X , with an admissible metric. Let \bar{L}_0 be the same line bundle with the canonical metric: if X has genus ≥ 1 , this is the metric induced by an embedding of X into its Jacobian, if X is of genus 0, then \bar{L}_0 is the trivial metrized line bundle. The metrized line bundle $\bar{L} \otimes \bar{L}_0^{-1}$ is the trivial line bundle, together with an admissible metric which is given by a function f_v at the place v of F .

A formula of Faltings–Hriljac expresses $(\hat{c}_1(\bar{L})^2|X)$ as minus twice the Néron–Tate height of the point of J corresponding to L . More generally,

$$(\hat{c}_1(\bar{L})^2|X) = -2\hat{h}_{\text{NT}}([L]) + \sum_{v \in M(F)} \mathcal{D}(f_v),$$

where for each $v \in M(F)$,

$$\mathcal{D}(f_v) = \int_{X_v} f_v dd^c(f_v)$$

is the *Dirichlet energy* of f_v . This is a non-positive quadratic form which vanishes if and only if f_v is constant. For more details, I refer to [16] at archimedean places and [55] at ultrametric places. (When X has genus 0, $L \simeq \mathcal{O}_X$ and the term $\hat{h}_{\text{NT}}([L])$ has to be interpreted as 0.)

As a consequence, $(\hat{c}_1(\bar{L})^2|X) \leq 0$. Let us analyse the case of equality. Since they are nonpositive, all terms in the formula above have to vanish. Consequently, $[L]$ is a torsion point in the Jacobian, and all functions f_v are constant. We will say that some power of \bar{L} is constant

3.4.2. Proposition. — *Let F be a number field, let X be a projective smooth curve over F . Let \bar{L} and \bar{M} be two admissible metrized line bundles over X . Assume that $\deg(L) = \ell$, $\deg(M) = m$ are positive. and $(\hat{c}_1(\bar{L})^2|X) = (\hat{c}_1(\bar{M})^2|X) = 0$. Then, the essential minimum of $\bar{L} \otimes \bar{M}$ satisfies the following inequality:*

$$e(\bar{L} \otimes \bar{M}) \geq -\frac{1}{2(\ell+m)\ell m} (\hat{c}_1(m\bar{L} - \ell\bar{M})^2|X).$$

Moreover, the right hand side of this inequality is always nonnegative and vanishes if and only if some power of $\bar{L}^m \otimes \bar{M}^{-\ell}$ is constant.

Proof. — By Zhang’s inequality (see [19]), one has

$$e(\bar{L} + \bar{M}) \geq \frac{1}{2(\ell+m)} (\hat{c}_1(\bar{L} + \bar{M})^2|X).$$

Since $(\hat{c}_1(\bar{L})^2|X) = (\hat{c}_1(\bar{M})^2|X) = 0$ by assumption, we observe that

$$(\hat{c}_1(\bar{L} + \bar{M})^2|X) = 2(\hat{c}_1(\bar{L})\hat{c}_1(\bar{M})|X) = -\frac{1}{\ell m} (\hat{c}_1(m\bar{L} - \ell\bar{M})^2|X).$$

This shows the first claim.

Since mL and ℓM have the same degree, viz. ℓm , the rest of the proposition follows from the negativity properties of the height recalled above. \square

3.4.3. — Assume that (x_n) is a generic sequence of points such that $h_{\overline{L}}(x_n)$ tends to 0. By Theorem 3.3.3, $h_{\overline{M}}(x_n)$ converges to

$$\frac{1}{\ell}(\widehat{c}_1(\overline{L})\widehat{c}_1(\overline{M})|X).$$

Except when both lower bounds are zero, this is strictly bigger than the lower bound of the proposition, which is equal to

$$\frac{1}{\ell + m}(\widehat{c}_1(\overline{L})\widehat{c}_1(\overline{M})|X).$$

In other words, the greedy obvious method to find points of small height for $\overline{L} + \overline{M}$ that first minimizes the height $h_{\overline{L}}$, only works up to the factor $(\ell + m)/\ell > 1$.

3.4.4. AN EXAMPLE. — Let us give some explicit formulae for the lower-bound above, in some particular cases. We consider $X = \mathbf{P}^1$ over \mathbf{Q} and the metrized line bundle $\overline{\mathcal{O}}(1)_W$. Let φ and ψ be polynomials with integral coefficients, of degrees ℓ and m respectively; let us put $\overline{L} = \varphi^* \overline{\mathcal{O}}(1)_W$, $\overline{M} = \psi^* \overline{\mathcal{O}}(1)_W$. The line bundle $\overline{L}^m \otimes \overline{L}^{-\ell}$ is trivial and its metric is given by a family of functions (f_v) . Since φ and ψ have integral coefficients, $f_v = 0$ at all finite places. Moreover, since $g_{\overline{L}}(x) = \log \max(|\varphi(x)|, 1)$ and $g_{\overline{M}}(x) = \log \max(|\psi(x)|, 1)$ are the Green functions for the divisors $\ell[\infty]$ and $m[\infty]$ respectively, one has

$$f_{\infty}(x) = \log \frac{\max(|\varphi(x)|^m, 1)}{\max(|\psi(x)|^{\ell}, 1)}.$$

Then,

$$\mathrm{dd}^c f_{\infty} = \frac{m}{2\pi} \mathrm{dArg} \varphi(x) \wedge \delta_{|\varphi(x)|=1} - \frac{\ell}{2\pi} \mathrm{dArg} \psi(x) \wedge \delta_{|\psi(x)|=1}$$

From this, we deduce that

$$\begin{aligned} \mathcal{D}(f_{\infty}) &= \frac{\ell m}{2\pi} \left(\int_{|\psi(x)|=1} \log \max(|\varphi(x)|, 1) \mathrm{dArg} \psi(x) \right. \\ &\quad \left. + \int_{|\varphi(x)|=1} \log \max(|\psi(x)|, 1) \mathrm{dArg} \varphi(x) \right), \end{aligned}$$

the two others terms vanishing. In fact, Stokes's formula implies that the two terms within the parentheses in the previous formula are equal and we have

$$\mathcal{D}(f_{\infty}) = \frac{\ell m}{\pi} \int_{|\varphi(x)|=1} \log \max(|\psi(x)|, 1) \mathrm{dArg} \varphi(x).$$

The simplest case to study is for $\varphi(x) = x^{\ell}$. Then,

$$\mathcal{D}(f_{\infty}) = \frac{\ell m}{\pi} \int_0^{2\pi} \log \max(|\psi(e^{i\theta})|, 1) \mathrm{d}\theta$$

is $2\ell m$ times the logarithm of the variant $M^+(\psi)$ of the Mahler measure of ψ :

$$M^+(\psi) = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \max(|\psi(e^{i\theta})|, 1) \mathrm{d}\theta \right).$$

In fact, Jensen's formula implies that

$$M^+(\psi) = \exp\left(\frac{1}{(2\pi)^2} \int_0^{2\pi} \log\left|\psi(e^{i\theta_1}) - e^{i\theta_2}\right| d\theta_1 d\theta_2\right)$$

is the Mahler measure $M(\psi(x) - y)$ of the 2-variables polynomial $\psi(x) - y$.

Consequently, except for finitely many exceptions, any algebraic point $x \in \mathbf{P}^1(\overline{\mathbf{Q}})$ satisfies

$$\ell h(x) + h(\psi(x)) \geq \frac{1}{\ell + m} \log M(\psi(x) - y).$$

For $\ell = 1$ and $\psi(x) = 1 - x$, we obtain that up to finitely many exceptions,

$$h(x) + h(1 - x) \geq \frac{1}{2} \log M(1 - x - y) \approx 0.161538,$$

In that particular case, Zagier [60] has proved a much more precise result: except for 5 explicit points in \mathbf{P}^1 ,

$$h(x) + h(1 - x) \geq \frac{1}{2} \log\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.240606.$$

Observe also that if (x_j) is a sequence of points such that $h(x_j) \rightarrow 0$, Theorem 3.3.3 implies that $h(1 - x) \rightarrow \log M(1 - x - y) \approx 0.323076$.

3.4.5. APPLICATION TO DYNAMICAL SYSTEMS. — Let us assume that \overline{L} and \overline{M} are the metrized line bundles $\overline{\mathcal{O}(1)}_\varphi$ and $\overline{\mathcal{O}(1)}_\psi$ attached to rational functions φ and ψ of degrees d and e respectively, with $d \geq 2$ and $e \geq 2$. Let us write h_φ and h_ψ for the height relative to these metrized line bundles; we call them the canonical heights. The isometry $\varphi^* \overline{\mathcal{O}(1)}_\varphi \simeq \overline{\mathcal{O}(1)}_\varphi^d$ and the functorial properties of the height imply that for any $x \in \mathbf{P}^1(\overline{F})$, $h_\varphi(\varphi(x)) = dh_\varphi(x)$ and $h_\psi(\psi(x)) = eh_\psi(x)$. In particular, preperiodic points for φ (*i.e.*, points with finite forward orbit) satisfy $h_\varphi(x) = 0$. Moreover,

$$\begin{aligned} d^2(\widehat{c}_1(\overline{\mathcal{O}(1)}_\varphi)^2|\mathbf{P}^1) &= (\widehat{c}_1(\varphi^* \overline{\mathcal{O}(1)}_\varphi)^2|\mathbf{P}^1) \\ &= (\widehat{c}_1(\overline{\mathcal{O}(1)}_\varphi)^2|\varphi_* \mathbf{P}^1) \\ &= d(\widehat{c}_1(\overline{\mathcal{O}(1)}_\varphi)^2|\mathbf{P}^1), \end{aligned}$$

hence $(\widehat{c}_1(\overline{\mathcal{O}(1)}_\varphi)^2|\mathbf{P}^1) = 0$ since $d \neq 0, 1$. Similarly, preperiodic points of ψ satisfy $h_\psi(x) = 0$, and $(\widehat{c}_1(\overline{\mathcal{O}(1)}_\psi)^2|\mathbf{P}^1) = 0$.

In the arithmetic case, or over function fields over a finite field, Northcott's finiteness theorem implies easily that points x such that $h_\varphi(x) = 0$ are preperiodic for φ , and similarly for ψ . This is not true in general: for example, if φ is constant, all constant points have height 0 but only countably many of them are preperiodic; more generally isotrivial rational functions, *i.e.* rational functions which are constant after conjugacy by an automorphism of \mathbf{P}^1 will furnish counterexamples. A theorem of Baker [4] shows that the converse is true: if φ is not isoconstant, then a point of height zero is then preperiodic; the proof relies on a detailed analysis of a Green function relative to the diagonal on $\mathbf{P}^1 \times \mathbf{P}^1$. (In a more general context than the case of polarized dynamical

systems, Chatzidakis and Hrushovski [21] proved that the Zariski closure of the orbit of a point of canonical height zero is isoconstant.)

Let us show how Prop. 3.4.2 implies some of the results of Baker and DeMarco [5], and of Petsche, Szpiro and Tucker [49].

3.4.6. Proposition. — *In the geometric case, let us assume that ψ is non-isotrivial; if F is a function field over an infinite field, let us moreover assume that it is a polynomial. The following are then equivalent:*

- (1) *the heights h_φ and h_ψ coincide;*
- (2) *φ and ψ have infinitely many common preperiodic points;*
- (3) *the essential lowest bound of $h_\varphi + h_\psi$ is zero;*
- (4) *the equilibrium measures μ_φ and μ_ψ are equal at all places;*
- (5) *the metrized line bundles $\mathcal{O}(1)_\varphi$ and $\mathcal{O}(1)_\psi$ are isomorphic, up to a family of constants (c_v) such that $\prod c_v = 1$.*

Proof. — The arguments are more or less formal from Prop. 3.4.2; let us detail them anyway for the sake of the reader.

1) \Rightarrow 2). Like any rational map, φ has infinitely many preperiodic points in $\mathbf{P}^1(\overline{F})$, and they satisfy $h_\varphi(x) = 0$. If $h_\varphi = h_\psi$, then they also satisfy $h_\psi(x) = 0$. Under the assumptions of the proposition, they are preperiodic for ψ .

2) \Rightarrow 3) is obvious, for common preperiodic points of φ and ψ satisfy $h_\varphi(x) + h_\psi(x) = 0$.

3) \Rightarrow 4). By Prop. 3.4.2, the line bundle $\mathcal{O}(1)_\varphi - \mathcal{O}(1)_\psi$ has the constant metric at all places. In particular, the local measures μ_φ and μ_ψ coincide at all places.

4) \Rightarrow 5). Let s be a non zero global section of $\mathcal{O}(1)$. For any place v , $f_v = \log(\|s\|_{v,\varphi} / \|s\|_{v,\psi})$; one has $\mu_{v,\psi} - \mu_{v,\varphi} = \text{dd}^c f_v$, hence $\text{dd}^c f_v = 0$. By the maximum principle of [55], f_v is constant. Moreover,

$$0 = (\widehat{c}_1(\overline{\mathcal{O}(1)}_\psi)^2|X) = (\widehat{c}_1(\overline{\mathcal{O}(1)}_\varphi)^2|X) + \sum_v \log c_v = \sum_v \log c_v.$$

5) \Rightarrow 1). This is obvious. □

3.4.7. REMARKS. — 1) The restrictive hypotheses on ψ have only been used to establish the implication 1) \Rightarrow 2).

2) Of course, many other results can be established by the same reasoning, in particular the number field case of Theorem 1.2 of [5]. Let us also recall that the support of the equilibrium measure μ_φ is the Julia set $J(\varphi)$. If one can prove that $J(\varphi) \neq J(\psi)$ at some place, then none of the assertions of Prop. 3.4.6 can possibly hold. Similarly, Theorem 1.1 of that article can be seen as the conjunction of our proposition and an independent complex analytic study of generalized Mandelbrot sets (Prop. 3.3).

3) The main result of [58] is that a variant of the implication (4) \Rightarrow (5) also holds in a more general setting: two semi-positive metrics on a line bundle which define the same measure at a place v differ by multiplication by a constant. The given proof works for curves.

4) We also recall that an implication similar to (1) \Rightarrow (5) holds for general metrized line bundles on arithmetic varieties, as proven by [1]: if \overline{L} and \overline{M} are line bundles with adelic metrics such that $h_{\overline{L}} = h_{\overline{M}}$, then $\overline{L} \otimes \overline{M}^{-1}$ is torsion in the Arakelov Picard group $\overline{\text{Pic}}(X)$: the heights determine the metrics.

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